



**ENTIRE BLOW-UP SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS
AND SYSTEMS**

THESIS

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SYSTEMS

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Abstract

We examine two problems concerning semilinear elliptic equations. We consider single equations of the form $\Delta u = p(x)u^\alpha + q(x)u^\beta$ for $0 < \alpha \leq \beta \leq 1$ and systems $\Delta u = p(|x|)f(v)$, $\Delta v = q(|x|)g(u)$, both in Euclidean n -space, $n \geq 3$. These types of problems arise in steady state diffusion, the electric potential of some bodies, subsonic motion of gases, and control theory. For the single equation case, we present sufficient conditions on p and q to guarantee existence of nonnegative bounded solutions on the entire space. We also give alternative conditions that ensure existence of nonnegative radial solutions blowing up at infinity. Similarly, for systems, we provide conditions on p, q, f , and g that guarantee existence of nonnegative solutions on the entire space. The main requirement for f and g will be closely related to a growth requirement known as the Keller-Osserman condition. Further, we demonstrate the existence of solutions blowing up at infinity and describe a set of initial conditions that would generate such solutions. Lastly, we examine several specific examples numerically to graphically demonstrate the results of our analysis.

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Table of Contents

	Page
Abstract	iv
Acknowledgments.....	v
Table of Contents.....	vi
List of Figures	vii
List of Tables	viii
I Introduction	1-1
II Background.....	2-1
2.1 Single Equations.....	2-1
2.2 Systems.....	2-4
2.3 Preliminaries.....	2-7
III Single Equations	3-1
3.1 Preliminaries.....	3-1
3.2 Main Results.....	3-3
3.3 Examples	3-14
IV Systems.....	4-1
4.1 Preliminaries.....	4-1
4.2 Main Results.....	4-5
4.3 Examples	4-28
V Conclusion	5-1
5.1 Summary	5-1
5.2 Further Work	5-3
Bibliography	Bib-1
Vita.....	Vita-1

List of Figures

	Page
Figure 3-1. Numerical Solutions of (3.32) for Various Central Values.....	3-15
Figure 3-2. Numerical Solutions of (3.34) for Various Central Values.....	3-16
Figure 3-3. Numerical Solutions of (3.35) for Various Central Values.....	3-17
Figure 4-1. Entire Solution Existence Region for Central Values of (1.2).....	4-16
Figure 4-2. Numerical Solutions of (4.40) for Varying Central Values	4-30
Figure 4-3. Numerical Solutions of (4.40) for Central Values Near Boundary.....	4-30
Figure 4-4. Entire Solution Existence Region for Central Values of (4.40).....	4-31
Figure 4-5. Non-monotonic Function f of System (4.41)	4-32
Figure 4-6. Numerical Solutions of (4.41) for Central Values Near Boundary.....	4-33
Figure 4-7. Numerical Solutions of (4.41) for Varying Central Values	4-34
Figure 4-8. Existence Region for Central Values of (4.41)	4-34
Figure 4-9. Solutions Showing Set of Central Values for (4.41) is Not Convex.....	4-35

List of Tables

	Page
Table 2-1. Existence of Solutions for Multiple Term Single Equation (1.1).....	2-4
Table 5-1. Existence of Solutions for System (1.2)	5-3

ENTIRE BLOW-UP SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS AND SYSTEMS

I. Introduction

Our work is divided into two parts. First, we examine the semilinear elliptic equation

$$\Delta u = p(x)u^\alpha + q(x)u^\beta, \quad x \in R^n \quad (1.1)$$

for $n \geq 3$. This problem was the focus of the thesis by Smith [26] in which multiple existence and nonexistence results were obtained. We examine this problem only briefly in an effort to close a few remaining gaps. The arguments we present are similar to those used in [26] and the works referenced therein.

The second part of our study is dedicated to semilinear elliptic systems. Namely, we examine

$$\begin{cases} \Delta u = p(|x|)f(v), \\ \Delta v = q(|x|)g(u), \end{cases} \quad x \in \Omega \subseteq R^n \quad (1.2)$$

where again $n \geq 3$. Here we are restricting our problem to the radial case. Several authors have studied this system for monotonic f and g . We will consider a more general system where f and g may be non-monotonic. To our knowledge, no results exist for system (1.2) when f and g are not monotone. This generalization presents several unique challenges that we must address.

For both single equations and systems, we are primarily concerned with proving the existence of large solutions. A solution to (1.1) is *large* if $u(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. If $\Omega = R^n$, we require $u(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. The latter case is called an *entire* large solution. Similarly, a *large* solution to system (1.2) is a solution (u, v) such that $u(|x|) \rightarrow \infty$ and $v(|x|) \rightarrow \infty$ as $|x| \rightarrow \infty$, and an *entire* large solution is one such that $u(|x|) \rightarrow \infty$ and $v(|x|) \rightarrow \infty$ as $|x| \rightarrow \infty$.

The arguments developed in our analysis and the mathematical foundations of elliptic theory may be applied to multiple problems in a wide variety of technical fields. Elliptic equations similar to those we analyze here are related to steady-state reaction-diffusion, subsonic fluid flows, electric potentials of some bodies, and control theory.

For example, [17] describes a general stochastic diffusion process with feedback controls. The controls are to be designed so that the state of the system is constrained to some region. Finding optimal controls is then shown to be equivalent to finding large solutions for a second order semilinear elliptic equation. As another example, [8] models the steady state of non-linear heat conduction through a 2-component mixture with a system of semilinear elliptic equations similar to the systems we study here

These are just two examples of application. The mathematical methods and ideas we use and develop here are applicable in many scientific and engineering disciplines. Before we discuss preliminaries for our work, let us first examine results obtained by others in this field.

II. Background

Bieberbach [3], in 1916, was the first to study large solutions to the semilinear elliptic problem

$$\Delta u = f(u), \quad x \in \Omega \quad (2.1)$$

where $f(u) = e^u$. Since that time, many authors have studied related problems for single equations and systems. In this section we will present only the most recent and/or most relevant accomplishments which have led to our study.

2.1 Single Equations

In 1957, Keller [11] and Osserman [21] established necessary and sufficient conditions for the existence of solutions to (2.1) on bounded domains in \mathbb{R}^n . They showed that large solutions exist on Ω if and only if f satisfies

$$\int_1^\infty \left[\int_0^t f(s) ds \right]^{-1/2} dt < \infty. \quad (2.2)$$

This requirement is sometimes referred to as the Keller-Osserman condition and continues to be significant in current studies. Indeed, we will require several of our functions to satisfy this Keller-Osserman condition.

Bandle and Marcus [2] later examined the equation

$$\Delta u = p(x)f(u) \quad (2.3)$$

for f non-decreasing on $[0, \infty)$ and proved the existence of positive large solutions provided the function f satisfies (2.2) and p is continuous and strictly positive on $\bar{\Omega}$. Lair [12] showed the same results hold for (2.3) when p is allowed to vanish on large parts of Ω , including its boundary. Lair, Proano, and Wood [16] relaxed the monotonicity condition on f by requiring that there exist some nonnegative, nondecreasing, Hölder continuous function g and positive constants M and s_0 such that

$$g(s) \leq f(s) \leq Mg(s) \text{ for all } s \geq s_0. \quad (2.4)$$

We note that our work on systems is a similar generalization; however, the arguments required for our analysis are very different from those used in the single equation case.

Many authors have examined more specific forms of (2.3). The equation

$$\Delta u = p(x)u^\gamma \quad (2.5)$$

has been of particular interest. Cheng and Ni [4] considered the superlinear case ($\gamma > 1$) and proved that (2.5) has large solutions on bounded domains provided p is strictly positive on $\partial\Omega$. Lair and Wood [14] generalized this to allow p to vanish on large portions of Ω including its boundary. They also showed the existence of an entire large solution to (2.5) provided that

$$\int_0^\infty r \max_{|x|=r} p(x) dr < \infty. \quad (2.6)$$

This is a weaker condition compared to the requirements in [4].

Fewer results are known for the sublinear case ($0 < \gamma \leq 1$) of (2.5). In [15], Lair and Wood proved that entire large radial solutions of (2.5) exist if and only if

$$\int_0^\infty rp(r)dr = \infty. \quad (2.7)$$

They also demonstrated that for a bounded domain Ω , (2.5) has no positive large solution when p is continuous in $\bar{\Omega}$. In addition to large solutions, Lair and Wood considered entire bounded solutions. They proved entire bounded solutions of (2.5) exist when (2.7) holds and p is locally Hölder continuous. Also, nonnegative, entire bounded solutions do not exist for (2.5) when

$$\int_0^\infty r \min_{|x|=r} p(x)dr = \infty. \quad (2.8)$$

Smith [26] then considered (1.1), which is a multi-term adaptation of the single term equation (2.5). For the superlinear ($1 < \alpha \leq \beta$) and mixed ($0 < \alpha \leq 1 < \beta$) cases, results comparable to those for single equations in [14] were obtained. In the sublinear case, Smith [26] proved the existence of entire large radial solutions and showed the existence of entire bounded solutions for the nonradial problem. However, both of these proofs require $0 < \alpha \leq \beta < 1$. The proofs do not hold when $\beta = 1$. Table 2-1 is a summary of the results from [26]. Our work on single equations removes condition (f) from the table, thereby closing the gap for $\beta = 1$.

Table 2-1. Existence of Solutions for Multiple Term Single Equation (1.1)

$\Delta u = p(x)u^\alpha + q(x)u^\beta$								
	Superlinear/Mixed $0 < \alpha \leq \beta, \beta > 1$		Sublinear $0 < \alpha \leq \beta \leq 1$			Sublinear $0 < \alpha \leq \beta \leq 1$		
	Bounded Domain $\Omega \subset \mathbb{R}^n$	Entire Domain $\Omega = \mathbb{R}^n$	Bounded Domain $\Omega \subset \mathbb{R}^n$	Entire Domain $\Omega = \mathbb{R}^n$		Entire Domain $\Omega = \mathbb{R}^n$		
Large Solutions/ Requirements	Yes a	Yes a,b	No	Yes c,d,f		Bounded Solutions	Yes b,f	No e
a) p, q are c-positive (see Definition 4-2) b) $\int_0^\infty r \max_{ x =r} p(x) dr < \infty$ and $\int_0^\infty r \max_{ x =r} q(x) dr < \infty$ c) radial case only: $p(x) = p(x)$ and $q(x) = q(x)$ d) $\int_0^\infty r \max_{ x =r} p(x) dr = \infty$ or $\int_0^\infty r \max_{ x =r} q(x) dr = \infty$ e) $\int_0^\infty r \min_{ x =r} p(x) dr = \infty$ or $\int_0^\infty r \min_{ x =r} q(x) dr = \infty$ f) $\beta \neq 1$								

2.2 Systems

Next we shall discuss the background of semilinear elliptic systems of the type given in (1.2). While systems are a natural extension of single equations and occur in many of the same areas of application, not all the methods employed to study them carry over. For instance, we do not have a meaningful maximum principle for systems. Additionally, the barrier method, which we present later, is a common method for proving the existence of solutions for single equations. However, a similar method for systems exists only when rather restrictive conditions are placed on the growths of the given functions. For example, if we consider systems built from equations comparable to (2.1), we have

$$\begin{cases} \Delta u = f(u, v), \\ \Delta v = g(u, v). \end{cases}$$

A barrier method similar to that for single equations exists only when f is monotonic in v , and g is monotonic in u . Still, this area has been well studied by several authors. Most related to our study, are the works of Lair and Wood [13], Cirstea and Radelescu [6], and Peng and Song [22].

Lair and Wood [13] examined the system

$$\begin{cases} \Delta u = p(|x|)v^\alpha, \\ \Delta v = q(|x|)u^\beta, \end{cases} \quad x \in \Omega \subseteq \mathbb{R}^n \quad (2.9)$$

where $n \geq 3$. They proved the existence of entire nonnegative solutions and characterized the set of central values, $S \subset \mathbb{R}^+ \times \mathbb{R}^+$, for these solutions. By central values, we mean $(u(0), v(0))$ where u, v are solutions of (2.9). Lair and Wood proved S is closed, bounded, and convex. They also further geometrically characterized this set by describing bounds for S . Finally, they proved that large solutions exist for central values that lie in the closure of $\{(a, b) \cap \partial S : a, b \neq 0\}$. We note that for $\alpha, \beta > 1$, they required

$$\int_0^\infty tp(t) < \infty, \quad \int_0^\infty tq(t) < \infty. \quad (2.10)$$

Both Peng and Song [22] and Cirstea and Radulescu [6] generalized their work to show comparable results for system (1.2) where $f, g \in C[0, \infty)$ are nonnegative monotonic functions. The approaches taken in each paper were slightly different, and each pair of authors required different additional constraints on f and g . In [6] several cases were considered. When

$$\lim_{t \rightarrow \infty} \frac{g(cf(t))}{t} = 0 \text{ for all } c > 0, \quad (2.11)$$

system (1.2) has entire solutions, all of which are bounded when (2.10) holds. Further, entire solutions exist and are large when neither inequality in (2.10) hold. Both papers then considered (1.2) where the nonnegative functions $p, q \in C[0, \infty)$ satisfy (2.10), and functions $f, g \in C[0, \infty)$ are nondecreasing, satisfy the Keller-Osserman condition (2.2), and

$$f(0) = g(0) = 0, \quad f(s) > 0, \quad g(s) > 0 \text{ for } s > 0. \quad (2.12)$$

In [22], these functions were further required to be convex, while in [6], there were additional conditions $f, g \in C^1[0, \infty)$ and

$$\liminf_{s \rightarrow \infty} \frac{f(s)}{g(s)} = \sigma > 0. \quad (2.13)$$

In these cases, both pairs of authors showed existence of entire large solutions and characterized the set of central values for which such solutions exist.

Our results on systems further generalize this problem by allowing f and g to be non-monotonic. Instead, we will require the function G , given by

$$G(s) = \min \left\{ \min_{s \leq t} f(t), \min_{s \leq t} g(t) \right\}, \quad 0 \leq s, \quad (2.14)$$

to satisfy the Keller-Osserman condition in (2.2). Note this automatically implies f and g must satisfy this condition as well.

Now we discuss preliminary theory that will be necessary for both single equations and systems.

2.3 Preliminaries

For readability, we shall reserve the terms lemma, theorem, and corollary for results which we justify in our work. Any result taken directly from another reference shall be labeled as Preliminary X-X.

The Arzela-Ascoli Theorem is our main tool for examining systems; however, it is also needed for a single equation result. Here we give the Arzela-Ascoli Theorem and several necessary definitions.

Definition 2-1 A subset K of a normed space X is called precompact (sometimes called relatively compact or conditionally compact) if its closure \bar{K} is compact in the norm topology of X .

Preliminary 2-1 (Theorem 1.34 of [1]) (Arzela-Ascoli Theorem) Let Ω be a bounded domain in \mathbb{R}^n . A subset K of $C(\bar{\Omega})$ is precompact in $C(\bar{\Omega})$ if

- (i) There exists $M \geq 0$ such that $|\phi(x)| \leq M$ for every $\phi \in K$ and $x \in \Omega$. That is, K is uniformly bounded.
- (ii) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|\phi(x) - \phi(y)| < \varepsilon$ for all $\phi \in K$, $x, y \in \Omega$, and $|x - y| < \delta$. That is, K is equicontinuous.

To apply the Arzela-Ascoli Theorem, we will often show that a sequence of functions is both uniformly bounded and equicontinuous. We will need to accomplish this for several different sequences throughout our arguments. However, these sequences will have similarities, and creating the following lemma will greatly streamline our work in later sections. This lemma will allow us to conclude a particular sequence is both uniformly bounded and equicontinuous only by showing uniform boundedness.

Lemma 2-1 Let $\{u_k\}$ be a sequence of functions of the form

$$u_k(r) = a_k + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) f(v_k(s)) ds dt, \quad r \in [0, R]. \quad (2.15)$$

where $a_k \in \mathbb{R}^+$, $p, f \in C[0, \infty)$ are nonnegative, and $\{v_k\}$ is an arbitrary sequence of nonnegative continuous functions on $[0, R]$. If the sequence $\{v_k\}$ is uniformly bounded on $[0, R]$, then $\{u_k\}$ is equicontinuous on $[0, R]$.

Proof. Since $\{v_k\}$ is uniformly bounded, there exists M such that $v_k(r) \leq M$ for all k and all $r \in [0, R]$. Then we have

$$\begin{aligned}
u'_k(r) &= r^{1-n} \int_0^r s^{n-1} p(s) f(v_k(s)) ds \\
&\leq r^{1-n} r^{n-1} \int_0^r p(s) f(v_k(s)) ds \\
&\leq \int_0^r p(s) \max_{0 \leq t \leq M} (f(t)) ds \\
&\leq \max_{0 \leq t \leq M} (f(t)) \int_0^R p(s) ds \\
&\equiv C < \infty.
\end{aligned}$$

Also, we clearly have $0 \leq u'_k(r)$. Thus, $\{u_k\}$ is equicontinuous on $[0, R]$, and our proof is complete. □

These are the basic ideas needed for both the single equation and system cases. Now we turn our attention to preliminaries required exclusively for our single equation arguments.

III. Single Equations

3.1 Preliminaries

We begin by presenting a result on the barrier method or upper-lower solution approach. This method is well-known for equations on bounded regions (see Theorem 2.3.1 of Sattinger [24]). However, we wish to apply the barrier method on the unbounded domain \mathbb{R}^n . We therefore use the following result (Preliminary 3-1) from Shaker [25]. We also provide definitions related to this result.

Definition 3-1 Let f be a function defined on an open set $\Omega \subseteq \mathbb{R}^n$. For $0 < \alpha \leq 1$, we say that f is Hölder continuous with exponent α , written $f \in C^\alpha$ on Ω , if for every $x, y \in \Omega$, there exists a nonnegative constant C such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha.$$

When $\alpha = 1$, we say f is Lipschitz continuous. Further, we say that f is locally Hölder (Lipschitz) continuous on $\Omega \subseteq \mathbb{R}^n$ when for every $x \in \Omega$, there exists an open ball $B(x, r)$ such that f restricted to $B(x, r)$ is Hölder (Lipschitz) continuous.

Preliminary 3-1 (Lemma 3 of [25]) (Lemma on Barrier Method) Let $u_1, u_2 : \mathbb{R}^n \rightarrow \mathbb{R}$,

$u_1(x) \geq u_2(x)$ for all $x \in \mathbb{R}^n$ be such that

$$\begin{aligned} Lu_1 + f(x, u_1) &\leq 0, \\ Lu_2 + f(x, u_2) &\geq 0, \end{aligned}$$

where f is locally Hölder continuous in (x, u) and locally Lipschitz in u , and L is an elliptic operator of second order. Then there exists a solution u of $Lu + f(x, u) = 0$ with $u_1 \geq u \geq u_2$.

The Laplacian is a second order elliptic operator, and therefore Preliminary 3-1 applies directly to equation (1.1).

Definition 3-2 Given Preliminary 3-1, we call u_1 an upper solution and u_2 a lower solution to $Lu + f(x, u) = 0$.

As mentioned with the Arzela-Ascoli Theorem, we will often be working with sequences of functions. In many of our proofs, we will be attempting to put bounds on these functions. One useful inequality for doing so is Gronwall's Inequality or Gronwall's Lemma.

Preliminary 3-2 (Theorem 1.5.7 of [10]) (Gronwall's Inequality) Let $t_0 \leq t \leq t_1$. Let $\psi(t)$ and $\phi(t)$ be continuous functions such that $\psi(t) \geq 0$ and

$$\phi(t) \leq K + \int_{t_0}^t \psi(s)\phi(s)ds$$

holds for $t_0 \leq t \leq t_1$ and K a constant. Then

$$\phi(t) \leq K \exp \left(\int_{t_0}^t \psi(s)ds \right)$$

for $t_0 \leq t \leq t_1$.

3.2 Main Results

We now present our main results for single equations. Our first result is an extension of Theorem 22 of [26] which required $0 < \alpha \leq \beta < 1$. We adopt a similar argument to obtain results for $0 < \alpha \leq \beta \leq 1$.

Theorem 3-1 Suppose p and q are nonnegative locally Hölder continuous in \mathbb{R}^n and

$$\int_0^\infty r \max_{|x|=r} p(x) dr < \infty, \quad \int_0^\infty r \max_{|x|=r} q(x) dr < \infty. \quad (3.1)$$

Then (1.1) has a nonnegative nontrivial entire bounded solution in \mathbb{R}^n if $0 < \alpha \leq \beta \leq 1$.

Proof. Since [26] shows the case for $0 < \alpha \leq \beta < 1$, we will assume $0 < \alpha \leq \beta = 1$. Define

$$\theta(r) = \max_{|x|=r} \{p(x), q(x)\}. \quad (3.2)$$

We will construct upper and lower solutions for (1.1) by examining the equation

$$\Delta z = \theta(r)(z^\alpha + z^\beta) = \theta(r)(z^\alpha + z), \quad r = |x| \in \mathbb{R}^n. \quad (3.3)$$

Since this equation is radial, we can rewrite the Laplacian in radial form. Doing so, equation (3.3) is equivalent to the ordinary differential equation

$$z''(r) + \frac{n-1}{r} z'(r) = \theta(r)(z^\alpha + z), \quad r \in [0, \infty). \quad (3.4)$$

Multiplying each side of (3.4) by r^{n-1} yields

$$r^{n-1} z''(r) + r^{n-1} \frac{n-1}{r} z'(r) = \left(r^{n-1} z'(r) \right)' = r^{n-1} \theta(r)(z^\alpha + z).$$

Integrating twice, we obtain

$$z(r) = c + \int_0^r t^{1-n} \int_0^t s^{n-1} \theta(s)(z^\alpha + z) ds dt, \quad (3.5)$$

where $c = z(0) \geq 0$ is our central value. Therefore a solution to (3.3) is any fixed point of the operator $T : C[0, \infty) \rightarrow C[0, \infty)$ defined by

$$Tz(r) = c + \int_0^r t^{1-n} \int_0^t s^{n-1} \theta(s)(z^\alpha + z) ds dt, \quad 0 \leq r. \quad (3.6)$$

Note the integration in this operator implies a fixed point $z \in C[0, \infty)$ is in $C^2[0, \infty)$, making z a classical solution to (3.3). We will show there exists such a fixed point $Tz = z$ and that z is bounded.

Let $z_0(r) = c$ for all $r \geq 0$, and define the sequence

$$z_k(r) = Tz_{k-1}(r) = c + \int_0^r t^{1-n} \int_0^t s^{n-1} \theta(s)(z_{k-1}^\alpha + z_{k-1}) ds dt \quad (3.7)$$

for $k = 1, 2, \dots$. Induction shows this sequence is increasing. Clearly

$$z_0(r) = c \leq c + \int_0^r t^{1-n} \int_0^t s^{n-1} \theta(s) (z_0^\alpha + z_0) ds dt = z_1(r).$$

Then, assuming $z_k \leq z_{k+1}$, we have

$$\begin{aligned} z_{k+1}(r) &= c + \int_0^r t^{1-n} \int_0^t s^{n-1} \theta(s) (z_k^\alpha + z_k) ds dt \\ &\leq c + \int_0^r t^{1-n} \int_0^t s^{n-1} \theta(s) (z_{k+1}^\alpha + z_{k+1}) ds dt \\ &= z_{k+2}(r). \end{aligned}$$

Hence $\{z_k\}$ is increasing. We now show $\{z_k\}$ is uniformly bounded. Integrating, (3.7) becomes

$$\begin{aligned} z_k(r) &= c + \int_0^r t^{1-n} \int_0^t s^{n-1} \theta(s) (z_{k-1}^\alpha + z_{k-1}) ds dt \\ &\leq c + \int_0^r t^{1-n} \int_0^t s^{n-1} \theta(s) (z_k^\alpha + z_k) ds dt \\ &= c + \frac{1}{2-n} \left[t^{2-n} \int_0^t s^{n-1} \theta(s) (z_k^\alpha + z_k) ds \right] \Big|_{t=0}^{t=r} - \int_0^r \frac{1}{2-n} t^{2-n} t^{n-1} \theta(t) (z_k^\alpha + z_k) dt. \end{aligned} \tag{3.8}$$

By L'Hopital's rule, we have

$$\begin{aligned} \lim_{t \rightarrow 0} \left[t^{2-n} \int_0^t s^{n-1} \theta(s) (z_k^\alpha + z_k) ds \right] &= \lim_{t \rightarrow 0} \left[\int_0^t s^{n-1} \theta(s) (z_k^\alpha + z_k) ds / t^{n-2} \right] \\ &= \lim_{t \rightarrow 0} \left[t^{n-1} \theta(t) (z_k^\alpha + z_k) / (n-2) t^{n-3} \right] \\ &= \lim_{t \rightarrow 0} \left[t^2 \theta(t) (z_k^\alpha + z_k) / (n-2) \right] \\ &= 0. \end{aligned} \tag{3.9}$$

Thus (3.8) and (3.9) imply

$$\begin{aligned}
z_k(r) &\leq c + \frac{1}{2-n} \left[t^{2-n} \int_0^t s^{n-1} \theta(s) (z_k^\alpha + z_k) ds \right] \Big|_{t=0}^{t=r} - \int_0^r \frac{1}{2-n} t^{2-n} t^{n-1} \theta(t) (z_k^\alpha + z_k) dt \\
&= c + \frac{1}{2-n} r^{2-n} \int_0^r s^{n-1} \theta(s) (z_k^\alpha + z_k) ds - \int_0^r \frac{t}{2-n} \theta(t) (z_k^\alpha + z_k) dt \\
&= c + \frac{1}{n-2} \left[\int_0^r t \theta(t) (z_k^\alpha + z_k) dt - r^{2-n} \int_0^r s^{n-1} \theta(s) (z_k^\alpha + z_k) ds \right] \\
&\leq c + \frac{1}{n-2} \int_0^r t \theta(t) (z_k^\alpha + z_k) dt.
\end{aligned} \tag{3.10}$$

We will consider the domain of z_k in two intervals. Notice that

$$z'_k(r) = r^{1-n} \int_0^r s^{n-1} \theta(s) (z_{k-1}^\alpha + z_{k-1}) ds \geq 0. \tag{3.11}$$

Thus, we may choose r_k such that

$$\begin{aligned}
z_k(r) &\leq 1 \quad \text{for } r \in [0, r_k], \\
z_k(r) &\geq 1 \quad \text{for } r \in [r_k, \infty).
\end{aligned} \tag{3.12}$$

It is possible that $r_k = 0$ or $r_k = \infty$. Indeed, since $z'_k(r) \geq 0$, we see from (3.12) and (3.5) that if

our central value $c > 1$, then $r_k = 0$. If $0 \leq c < 1$, then either $r_k = \infty$ with $z_k(r) \leq 1$ for all $r \geq 0$

or $r_k < \infty$ with $z_k(r_k) = 1$. We are attempting to bound $\{z_k\}$, and therefore $r_k = \infty$ is trivial.

Therefore if we consider $r_k < \infty$, and split our domain, (3.10) becomes

$$\begin{aligned}
z_k(r) &\leq c + \frac{1}{n-2} \int_0^{r_k} t\theta(t)(z_k^\alpha + z_k)dt + \frac{1}{n-2} \int_{r_k}^r t\theta(t)(z_k^\alpha + z_k)dt \\
&\leq c + \frac{2}{n-2} \int_0^{r_k} t\theta(t)dt + \frac{1}{n-2} \int_{r_k}^r t\theta(t)(2z_k)dt \\
&\leq c + \frac{2}{n-2} \int_0^\infty t\theta(t)dt + \frac{2}{n-2} \int_{r_k}^r t\theta(t)z_k dt \\
&= R + \frac{2}{n-2} \int_{r_k}^r t\theta(t)z_k dt
\end{aligned}$$

where R is a constant because (3.1) implies

$$\int_0^\infty t\theta(t)dt < \infty. \quad (3.13)$$

Finally, Gronwall's inequality (Preliminary 3-2) yields

$$\begin{aligned}
z_k(r) &\leq R + \frac{2}{n-2} \int_{r_k}^r t\theta(t)z_k dt \\
&\leq R \exp\left(\frac{2}{n-2} \int_{r_k}^r t\theta(t)dt\right) \\
&\leq R \exp\left(\frac{2}{n-2} \int_0^\infty t\theta(t)dt\right) \\
&= M < \infty
\end{aligned}$$

which is finite due to (3.13). We have shown that $\{z_k\}$ is a uniformly bounded monotonic sequence. Therefore it converges pointwise to some z . Further, since $\{z_k\}$ is in the form of (2.15), Lemma 2-1 implies that $\{z_k\}$ is also equicontinuous. Pointwise convergence and equicontinuity imply uniform convergence, and thus $z \in C^2[0, \infty)$. We have a fixed point of

(3.6) and bounded solution of (3.3). Finally, we show this function z and its bound M form upper and lower solutions to (1.1).

Let $u_1 \equiv M \geq z \equiv u_2$. Clearly,

$$\begin{aligned}\Delta u_1 &= 0 \leq pu_1^\alpha + qu_1, \\ \Delta u_2 &= \theta(u_2^\alpha + u_2) \geq pu_2^\alpha + qu_2,\end{aligned}$$

so

$$\begin{aligned}\Delta u_1 - (pu_1^\alpha + qu_1) &\leq 0, \\ \Delta u_2 - (pu_2^\alpha + qu_2) &\geq 0.\end{aligned}$$

Hence, there exists a positive nontrivial entire bounded solution u of (1.1) such that

$M = u_1 \geq u \geq u_2 = z$ by the Barrier Method (Preliminary 3-1). □

We will also consider Theorem 21 in [26] for $0 < \alpha \leq \beta \leq 1$. This again fills the gap for when $\beta = 1$.

Theorem 3-2 Suppose $0 < \alpha \leq \beta \leq 1$ and that $p(x) = p(|x|) \in C(\mathbb{R})$ and $q(x) = q(|x|) \in C(\mathbb{R})$ such that p and q are nonnegative. Then equation (1.1) has an entire large positive solution if and only if

$$\int_0^\infty rp(r)dr = \infty \text{ or } \int_0^\infty rq(r)dr = \infty. \quad (3.14)$$

Proof. Again, since Smith [26] proved this result for $0 < \alpha \leq \beta < 1$, we will consider

$0 < \alpha \leq \beta = 1$. An argument for necessity identical to that in Theorem 21 of [26] will work for $\beta = 1$. We are left to show sufficiency.

Radial solutions of (1.1) satisfy the ordinary differential equation

$$u''(r) + \frac{n-1}{r}u'(r) = p(|x|)u^\alpha + q(|x|)u^\beta. \quad (3.15)$$

As in Theorem 3-1, it follows that (3.15) has nonnegative solutions if the operator

$T : C[0, \infty) \rightarrow C[0, \infty)$ defined by

$$Tu(r) = c + \int_0^r t^{1-n} \int_0^t s^{n-1} (p(s)u^\alpha + q(s)u) ds dt \quad (3.16)$$

has a fixed point in $C[0, \infty)$ where $u(0) = c \geq 0$ is our central value. We begin by showing that

(3.16) has a fixed point in $C[0, R]$ for arbitrary $0 < R < \infty$. Similar to our previous proof, we let

$u_0 = c$ and define the sequence

$$u_{k+1} = Tu_k(r) = c + \int_0^r t^{1-n} \int_0^t s^{n-1} (p(s)u_k^\alpha + q(s)u_k) ds dt, \quad (3.17)$$

$k = 1, 2, \dots$. The same induction argument from Theorem 3-1 shows the sequence $\{u_k\}$ is non-

decreasing. We will now prove that $\{u_k\}$ is uniformly bounded and equicontinuous on

$0 \leq r \leq R$.

Define

$$\theta(s) = \max \{ p(s), q(s) \}. \quad (3.18)$$

Integrating as we did in (3.8) and (3.10), we have

$$u_{k+1}(r) \leq c + \frac{1}{n-2} \int_0^r s \theta(s) (u_k^\alpha + u_k) ds, \quad 0 \leq r \leq R. \quad (3.19)$$

Also similar to the previous proof, we have $u'_k \geq 0$, so we may choose r_k such that

$$\begin{aligned} u_k(r) &\leq 1 \quad \text{for } r \in [0, r_k], \\ u_k(r) &\geq 1 \quad \text{for } r \in [r_k, R]. \end{aligned} \quad (3.20)$$

We are attempting to obtain a bound, and thus we need only consider r_k finite and $r \geq r_k$.

Splitting our integral from (3.19) into two pieces, we have by (3.20)

$$\begin{aligned} u_{k+1}(r) &\leq c + \frac{1}{n-2} \int_0^r s \theta(s) (u_k^\alpha + u_k) ds \\ &= c + \int_0^{r_k} \frac{1}{n-2} s \theta(s) (u_k^\alpha + u_k) ds + \int_{r_k}^r \frac{1}{n-2} s \theta(s) (u_k^\alpha + u_k) ds \\ &\leq c + \int_0^{r_k} \frac{2}{n-2} s \theta(s) ds + \int_{r_k}^r \frac{1}{n-2} s \theta(s) (2u_k) ds \\ &= c + \int_0^{r_k} h(s) ds + \int_{r_k}^r h(s) u_k ds \end{aligned}$$

where $h(s) = 2s\theta(s)/(n-2)$. Since $\{u_k\}$ is an increasing sequence of increasing functions, we

must have $r_{k+1} \leq r_k$ for all $k = 0, 1, \dots$. Thus, $r_0 \geq r_k$ for all k which yields

$$\begin{aligned}
u_{k+1}(r) &\leq c + \int_0^{r_k} h(s) ds + \int_{r_k}^r h(s) u_k ds \\
&\leq c + \int_0^{r_0} h(s) ds + \int_0^r h(s) u_k ds \\
&= W + \int_0^r h(s) u_k ds.
\end{aligned} \tag{3.21}$$

We now use induction to show

$$u_{k+1}(r) \leq W \exp\left(\int_0^r h(s) ds\right). \tag{3.22}$$

Clearly we have

$$u_0 = c \leq \left[c + \int_0^{r_0} h(s) ds \right] \exp\left(\int_0^r h(s) ds\right) = W \exp\left(\int_0^r h(s) ds\right).$$

Now, assume that (3.22) is true for arbitrary k . Then for $k+1$, (3.21) implies,

$$\begin{aligned}
u_{k+1} &\leq W + \int_0^r h(s) u_k(s) ds \\
&\leq W + \int_0^r h(s) W \exp\left(\int_0^s h(t) dt\right) ds \\
&= W + W \exp\left(\int_0^s h(t) dt\right) \Big|_0^r \\
&= W + W \exp\left(\int_0^r h(s) ds\right) - W \\
&= W \exp\left(\int_0^r h(s) ds\right).
\end{aligned}$$

Thus, (3.22) is true by induction. Finally, since each u_k is increasing, we arrive at

$$u_k(r) \leq W \exp\left(\int_0^r h(t)dt\right) \leq W \exp\left(\int_0^R h(t)dt\right) \equiv M_R, \quad 0 \leq r \leq R. \quad (3.23)$$

That is, $\{u_k\}$ is uniformly bounded on $r \in [0, R]$.

Notice that our sequence is of the same form as (2.15). Therefore $\{u_k\}$ is also equicontinuous on $r \in [0, R]$ by Lemma 2-1. Since $\{u_k\}$ is a monotonic, uniformly bounded, equicontinuous sequence of functions on $[0, R]$, $u_k \rightarrow u$ uniformly. That is $Tu = u$ for $r \in [0, R]$, and we have a fixed point of (3.16) in $C[0, R]$.

Next, we extend this result to show that (3.16) has a fixed point in $C[0, \infty)$. We do so using a diagonal argument similar to the argument in Theorem 1 of [15]. Define the sequence of fixed points $\{w_k\}$ by

$$Tw_k = w_k \text{ on } [0, k], \quad w_k \in C[0, k]. \quad (3.24)$$

Restricted on $[0, 1]$, $\{w_k\}$ is bounded by M_1 as given in (3.23). Using Lemma 2-1, we can also show it is equicontinuous on this interval. Thus, the Arzela-Ascoli Theorem (Preliminary 2-1) implies that there exists a subsequence, call it $\{w_k^1\}$, that converges uniformly on $[0, 1]$. Let

$$w_k^1 \rightarrow v_1 \text{ uniformly on } [0, 1] \text{ as } k \rightarrow \infty. \quad (3.25)$$

Similarly, the sequence $\{w_k^1\}$ is bounded and equicontinuous on the interval $[0, 2]$. Hence, it must contain a convergent subsequence $\{w_k^2\}$ that converges uniformly on $[0, 2]$. Let

$$w_k^2 \rightarrow v_2 \text{ uniformly on } [0, 2] \text{ as } k \rightarrow \infty. \quad (3.26)$$

Note that $\{w_k^2\} \subseteq \{w_k^1\} \subseteq \{w_k\}_{k=2}^\infty$. This implies $v_2 = v_1$ on $[0, 1]$. Continuing, we have

$\{w_k^n\} \subseteq \cdots \subseteq \{w_k^1\} \subseteq \{w_k\}_{k=n}^\infty$ and a sequence $\{v_k\}$ such that

$$\begin{aligned} v_k(r) &\in C[0, k] & k = 1, 2, \dots \\ v_k(r) &= v_1(r) & \text{for } r \in [0, 1] \\ v_k(r) &= v_2(r) & \text{for } r \in [0, 2] \\ &\vdots \\ v_k(r) &= v_{k-1}(r) & \text{for } r \in [0, k-1]. \end{aligned} \quad (3.27)$$

Thus, we obtain a sequence $\{v_k\}$ that converges to v on $[0, \infty)$ satisfying

$$v(r) = v_k(r) \text{ if } 0 \leq r \leq k. \quad (3.28)$$

This convergence is uniform on bounded sets, implying $v \in C[0, \infty)$. We therefore have our fixed point $Tv = v$ of (3.16) in $C[0, \infty)$, and equation (1.1) has an entire radial solution. We lastly must show that this solution is large.

We note the argument of Theorem 1 in [15] demonstrates that (3.14) implies

$$\int_0^\infty t^{1-n} \int_0^t s^{n-1} (p(s) + q(s)) ds dt = \infty. \quad (3.29)$$

Our solution satisfies

$$\begin{aligned}
u(r) &= c + \int_0^r t^{1-n} \int_0^t s^{n-1} [p(s)u^\alpha + q(s)u] ds dt \\
&\geq c + \min \{u^\alpha(0), u(0)\} \int_0^r t^{1-n} \int_0^t s^{n-1} [p(s) + q(s)] ds dt \\
&= c + \min \{u^\alpha(0), u(0)\} \int_0^r t^{1-n} \int_0^t s^{n-1} [p(s) + q(s)] ds dt.
\end{aligned} \tag{3.30}$$

Therefore, letting $r \rightarrow \infty$, (3.29) and (3.30) imply $\lim_{r \rightarrow \infty} u(r) = \infty$, and we have a large solution.

This concludes our proof. \square

The condition $\beta = 1$ remained an open problem because comparisons between $u_k^\alpha(r)$ and $u_k^\beta(r)$ are quite different depending whether or not $u_k(r) < 1$ or $u_k(r) > 1$. To resolve this issue, we simply split the domains of our functions into two intervals and carried out the comparisons separately.

3.3 Examples

In this section we present several examples of equations that satisfy the restrictions given in our analysis. We also solve several examples numerically. Note that Theorem 3-2 only applies to radial equations while Theorem 3-1 applies to more general nonradial problems. For simplicity, all our examples will be radial.

Examining Theorem 3-1, we must choose functions p, q that satisfy (3.1). That is, equation (1.1) must contain nonnegative functions $p, q \in C[0, \infty)$ that decay at faster than $1/s^2$ as $s \rightarrow \infty$. For example,

$$\Delta u = (|x| + 1)^{-3} u^\alpha + (|x| + 2)^{-4} u^\beta \quad \text{or} \quad \Delta u = 2^{-|x|} u^\alpha + 3^{-|x|} u^\beta \tag{3.31}$$

are two very simple equations that meet this requirement.

According to our analysis, for $0 < \alpha \leq \beta \leq 1$, entire bounded solutions exist for equations of this form. Choose $\alpha = 1/2$, $\beta = 1$, and consider

$$\Delta u = 2^{-|x|} u^{1/2} + 3^{-|x|} u. \quad (3.32)$$

Using a Runge-Kutta algorithm, we will solve this problem numerically. Figure 3-1 shows these solutions for a variety of central values as initial conditions. Indeed, we have bounded entire solutions.

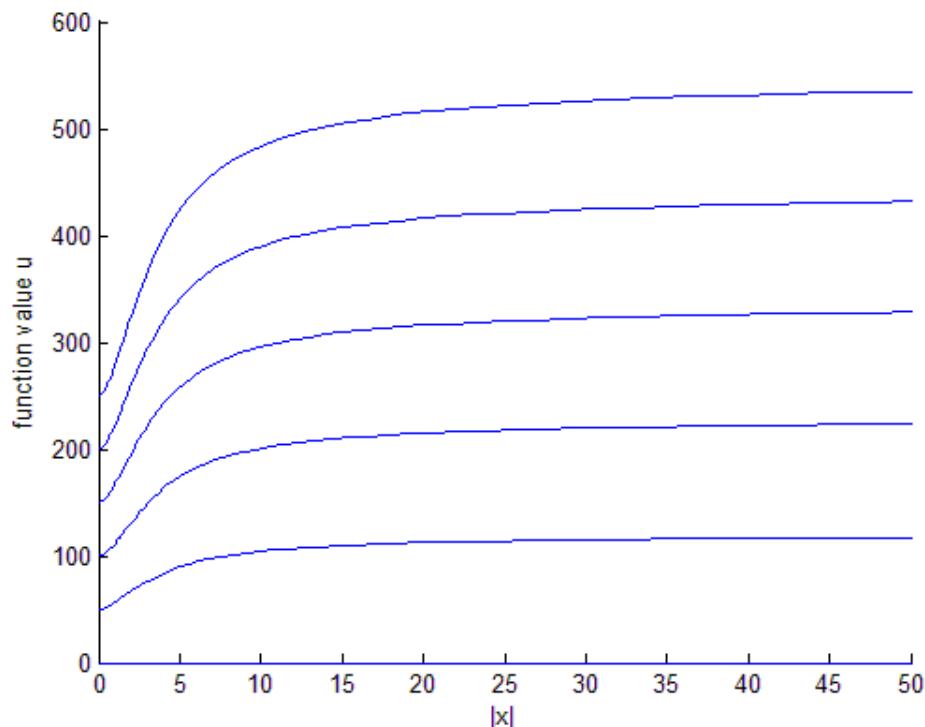


Figure 3-1. Numerical Solutions of (3.32) for Various Central Values

Next, we consider examples for Theorem 3-2. We now must have p, g satisfy (3.14). A few simple examples would be

$$\Delta u = |x| u^\alpha + |x|^2 u^\beta, \quad \Delta u = 2^{|x|} u^\alpha + 3^{|x|} u^\beta, \quad \Delta u = \left(1/(1+|x|)\right) u^\alpha + \left(1/(1+|x|)\right) u^\beta, \quad \text{or} \quad (3.33)$$

$$\Delta u = (1+|x|)^{-2} u^\alpha + (1+|x|)^{-2} u^\beta.$$

Again, we will choose to examine some of these equations numerically. Consider

$$\Delta u = \left(1/(1+|x|)\right) u^{1/2} + \left(1/(1+|x|)\right) u. \quad (3.34)$$

We examine a variety of central values between 0 and 100, and plot our numerical solutions in Figure 3-2.

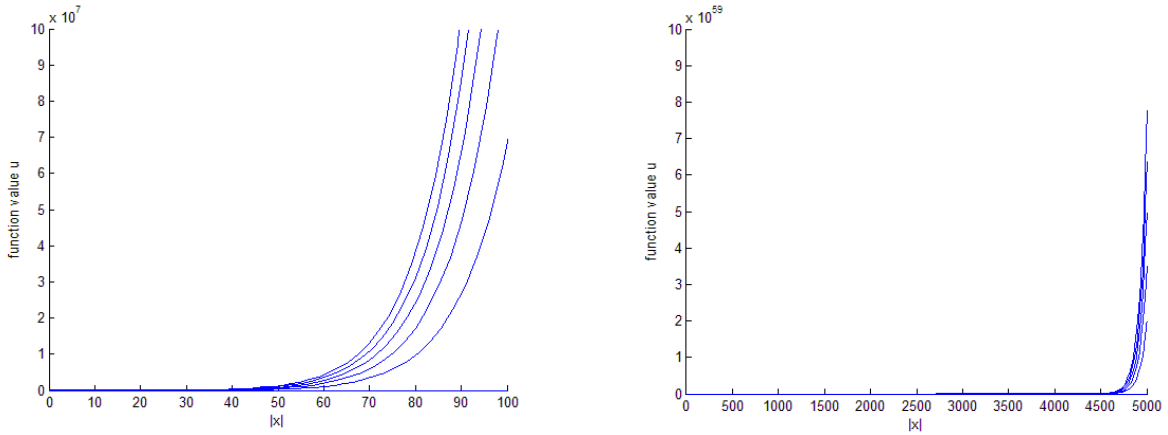


Figure 3-2. Numerical Solutions of (3.34) for Various Central Values

We show results for $0 \leq |x| \leq 100$ and clearly observe solutions grow quickly. We then show the same solutions in $0 \leq |x| \leq 5000$. As we examine larger values for $|x|$, our solutions continue to be defined. This is exactly what we would expect; our analysis guarantees these solutions are entire and large.

We will examine another interesting equation. Consider

$$\Delta u = (1+|x|)^{-2} u^\alpha + (1+|x|)^{-2} u^\beta. \quad (3.35)$$

Note for this example $p(r) = q(r) = 1/(1+r)^2$ satisfies (3.14) even though

$\lim_{r \rightarrow \infty} rp(r) = \lim_{r \rightarrow \infty} rq(r) = \lim_{r \rightarrow \infty} r/(1+r)^2 = 0$. The integrand decays to zero, but the value of the

integral in (3.14) still approaches infinity as $r \rightarrow \infty$. Solutions are shown for a variety of central values in Figure 3-3.

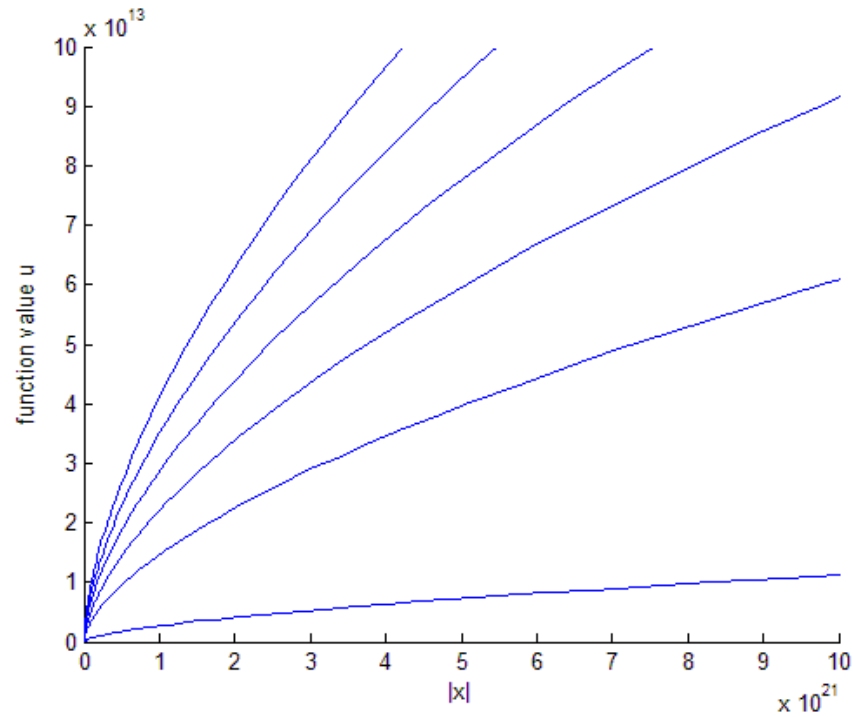


Figure 3-3. Numerical Solutions of (3.35) for Various Central Values

Our solutions grow much slower. From the plot, it is not clear that solutions are large. However, our analysis guarantees that every solution goes to infinity as $|x| \rightarrow \infty$.

IV. Systems

We now consider semilinear elliptic systems as given in (1.2). As with our section on single equations, we begin by presenting preliminary material essential for our study of systems.

4.1 Preliminaries

We first provide several definitions. We define a set of central values, what it means for a function to be circumferentially positive, and we construct important functions.

Definition 4-1 (Defined in Theorem 1 of [13]) Given system (1.2), we define the set of central values S as

$$S \equiv \{(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+ : u(0) = a, v(0) = b, \text{ and } (u, v) \text{ is an entire solution to (1.2)}\}. \quad (4.1)$$

Definition 4-2 (Condition A in [12]) A function p is said to be circumferentially positive (c-positive) on a domain Ω if for any $x_0 \in \Omega$ satisfying $p(x_0) = 0$, there exists a domain Ω_0 such that $x_0 \in \Omega_0$, $\overline{\Omega_0} \subset \Omega$, and $p(x) > 0$ for all $x \in \partial\Omega_0$.

Definition 4-3 Given system (1.2), we define

$$\begin{aligned} \psi(|x|) &= \min \{p(|x|), q(|x|)\}, \\ \phi(|x|) &= \max \{p(|x|), q(|x|)\}. \end{aligned} \quad (4.2)$$

We briefly note that these functions have some useful properties. Namely,

$\psi(|x|) \leq p(|x|) \leq \phi(|x|)$, $\psi(|x|) \leq q(|x|) \leq \phi(|x|)$ and ϕ, ψ are both nonnegative and c-

positive when p, q are nonnegative and not identically zero at infinity. Also, we note that when p, q satisfy (2.10), we have

$$\int_0^{\infty} t\psi(t)dt \leq \int_0^{\infty} t\phi(t)dt \leq \int_0^{\infty} t(p(t) + q(t))dt = \int_0^{\infty} tp(t)dt + \int_0^{\infty} tq(t)dt < \infty.$$

Thus ψ, ϕ also satisfy (2.10).

Definition 4-4 Given system (1.2), we define

$$\begin{aligned} G(s) &= \min \left\{ \min_{s \leq t} f(t), \min_{s \leq t} g(t) \right\}, & 0 \leq s, \\ H(s) &= \max \left\{ \max_{0 \leq t \leq s} f(t), \max_{0 \leq t \leq s} g(t) \right\}, & 0 \leq s. \end{aligned} \tag{4.3}$$

These functions have some very important characteristics as well. We demonstrate these properties in the following simple lemma.

Lemma 4-1 Let $f, g \in C[0, \infty)$ satisfy (2.12), and suppose G , shown in (4.3), satisfies the Keller-Osserman condition (2.2). Then G and H as given in (4.3) are continuous, nondecreasing, and satisfy

$$\begin{aligned} G(s) &\leq f(s) \leq H(s), \\ G(s) &\leq g(s) \leq H(s). \end{aligned} \tag{4.4}$$

Further, G and H satisfy (2.2) and (2.12).

Proof. It is clear G and H are continuous, nondecreasing, and satisfy (2.12) from definition. It is also clear that

$$\begin{aligned} G(s) &\leq f(s) \leq H(s), \\ G(s) &\leq g(s) \leq H(s). \end{aligned} \tag{4.5}$$

Function G satisfies the Keller-Osserman condition (2.2) by hypothesis. Then, since

$G(s) \leq H(s)$, we have

$$\int_1^\infty \left[\int_0^t H(s) ds \right]^{-1/2} dt \leq \int_1^\infty \left[\int_0^t G(s) ds \right]^{-1/2} dt < \infty.$$

Thus H satisfies the Keller-Osserman condition, and our proof is complete. \square

As we develop an argument for the existence of entire large solutions for system (1.2), we will first show that solutions exist on bounded domains for central values outside of S (given in (4.1)). In order to prove this, we will utilize a version of Schauder Fixed Point Theorem. Lemma 4-2 is the version we will apply and is a very simple consequence of this theorem.

Preliminary 4-1 (Corollary 11.2 of [9]) (Corollary of Schauder's Fixed Point Theorem) Let G be a closed convex set in a Banach space B , and let T be a continuous transformation of G into itself such that the image TG is precompact. Then T has a fixed point.

Definition 4-5 A linear transformation between two Banach spaces is called compact (sometimes called completely continuous) if the images of bounded sets are precompact.

Preliminary 4-2 (Theorem 5.24.2 of [19]) Any compact linear transformation between Banach spaces is continuous.

Lemma 4-2 Let G be a closed, convex, and bounded set in a Banach space B , and let T be a compact transformation that maps G into itself. Then T has a fixed point.

Proof. This follows directly from Preliminary 4-1, Definition 4-5, and Preliminary 4-2. \square

After using Lemma 4-2 to establish solutions on bounded domains, we will show that any bounded solution on such a domain may be extended to include a larger domain. We accomplish this extension with Caratheodory's Theorem. We now provide this theorem and related definitions.

Definition 4-6 Consider the equation

$$x' = f(t, x) \tag{4.6}$$

with initial condition $x(\tau) = \xi$. We say that ϕ is a solution to (4.6) in the extended sense on some interval I if $\phi'(t) = f(t, \phi(t))$ for all $t \in I$ except on a set of Lebesgue-measure zero and $\phi(\tau) = \xi$.

Preliminary 4-3 (Theorem 1.1 of Chapter 2 of [7]) (Caratheodory's Theorem) Consider the equation given in (4.6). Let a, b be constants, and define the rectangle

$R = \{(t, x) : |t - \tau| \leq a, |x - \xi| \leq b\}$ where (τ, ξ) is a fixed point in the (t, x) plane. Let f be defined on R . Suppose f is measurable in t for each fixed x , and continuous in x for each fixed t . If there exists a Lebesgue-integrable function m on the interval $|t - \tau| \leq a$ such that

$$|f(t, x)| \leq m(t), \quad (t, x) \in R,$$

then there exists a solution ϕ of (4.6) in the extended sense on some interval $|t - \tau| \leq \beta$, $(\beta > 0)$ satisfying $\phi(\tau) = \xi$.

Note that if f is continuous on R , then ϕ is a solution in the traditional sense. That is $\phi'(t)$ is continuous and $\phi'(t) = f(t, \phi(t))$ for all $t \in (\tau - \beta, \tau + \beta)$. We are now ready to present our main results for systems.

4.2 Main Results

The theorems and lemmas presented in this section build off one another. We begin by proving existence of entire solutions and work toward showing the existence of entire large solutions.

Theorem 4-1 Assume $p, q \in C(\mathbb{R}^n)$ are nonnegative, not identically zero at infinity, and satisfy (2.10). Let $f, g \in C[0, \infty)$ satisfy (2.12), and let G , given in (4.3), satisfy the Keller-Osserman condition (2.2). Then system (1.2) has infinitely many entire nonnegative solutions.

Proof. Radial solutions of system (1.2) are solutions to the ordinary differential system

$$\begin{cases} u''(r) + \frac{n-1}{r} u'(r) = p(r)f(v(r)), & r \geq 0, \\ v''(r) + \frac{n-1}{r} v'(r) = q(r)g(u(r)), & r \geq 0. \end{cases} \quad (4.7)$$

It follows that solutions to (1.2) are fixed points of the operator

$T(u, v) : C[0, \infty) \times C[0, \infty) \rightarrow C[0, \infty) \times C[0, \infty)$ defined by

$$T(u(r), v(r)) = \begin{cases} a + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s)f(v(s))dsdt, & r \geq 0, \\ b + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s)g(u(s))dsdt, & r \geq 0 \end{cases} \quad (4.8)$$

where $u(0) = a \geq 0$ and $v(0) = b \geq 0$ are the central values for the system. We will begin by

establishing a fixed point of (4.8) in $C[0, R] \times C[0, R]$ for arbitrary $R > 0$.

Define $u_0(r) = a$ and $v_0(r) = b$ for all $r \geq 0$, and consider the sequence

$$(u_k, v_k) = T(u_{k-1}, v_{k-1}) = \begin{cases} a + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s)f(v_{k-1}(s))dsdt, & r \geq 0, \\ b + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s)g(u_{k-1}(s))dsdt, & r \geq 0 \end{cases} \quad (4.9)$$

for $k = 1, 2, \dots$. Consider the single equation \bar{u} given by

$$\begin{aligned} \Delta \bar{u} &= \phi(|x|)H(\bar{u}), \quad x \in \mathbb{R}^n, \\ \bar{u} &\rightarrow \infty \quad \text{as } |x| \rightarrow \infty \end{aligned} \quad (4.10)$$

where ϕ and H are defined in (4.2) and (4.3) respectively. Recall that ϕ is nonnegative, c-positive, and satisfies (2.10). By Lemma 4-1, H satisfies the Keller-Osserman condition (2.2). These conditions guarantee such a \bar{u} exists by Theorem 2 of [12]. Further, this radial solution satisfies

$$\bar{u}(r) = \bar{u}(0) + \int_0^r t^{1-n} \int_0^t s^{n-1} \phi(s) H(\bar{u}(s)) ds dt, \quad r \geq 0 \quad (4.11)$$

where $\bar{u}(0) > 0$. Note this solution is non-decreasing. Indeed

$$\frac{d}{dr} \bar{u}(r) = r^{1-n} \int_0^r s^{n-1} \phi(s) H(\bar{u}(s)) ds \geq 0. \quad (4.12)$$

We shall choose our central values a and b such that $0 \leq a \leq \bar{u}(0)$ and $0 \leq b \leq \bar{u}(0)$. Next, we use induction to show each individual sequence in (4.9) is uniformly bounded on bounded sets.

Specifically, we will show $\max\{u_k(r), v_k(r)\} \leq \bar{u}(R)$ for all k and all $r \in [0, R]$.

Since H and \bar{u} are non-decreasing functions and $a, b \leq \bar{u}(0)$, we have

$$\begin{aligned} u_1(r) &= a + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) f(v_0(s)) ds dt \\ &\leq \bar{u}(0) + \int_0^r t^{1-n} \int_0^t s^{n-1} \phi(s) H(b) ds dt \\ &\leq \bar{u}(0) + \int_0^r t^{1-n} \int_0^t s^{n-1} \phi(s) H(\bar{u}(s)) ds dt \\ &= \bar{u}(r), \end{aligned}$$

and similarly $v_1(r) \leq \bar{u}(r)$. Now, assume $u_k, v_k \leq \bar{u}$. This implies

$$\begin{aligned}
u_{k+1}(r) &= a + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) f(v_k(s)) ds dt \\
&\leq \bar{u}(0) + \int_0^r t^{1-n} \int_0^t s^{n-1} \phi(s) H(v_k(s)) ds dt \\
&\leq \bar{u}(0) + \int_0^r t^{1-n} \int_0^t s^{n-1} \phi(s) H(\bar{u}(s)) ds dt \\
&= \bar{u}(r).
\end{aligned}$$

We similarly obtain $v_{k+1}(r) \leq \bar{u}(r)$, so $u_k, v_k \leq \bar{u}(r)$ by induction. Again, since \bar{u} is non-decreasing, we have the bound

$$\begin{aligned}
u_k(r), v_k(r) &\leq \bar{u}(r) \\
&\leq \bar{u}(R). \\
&\equiv M
\end{aligned} \tag{4.13}$$

for all $k \geq 1$ and all $r \in [0, R]$. We know $M < \infty$ because \bar{u} is entire.

Then, since each individual sequence $\{u_k\}, \{v_k\}$ is of the form (2.15), Lemma 2-1 guarantees that $\{u_k\}, \{v_k\}$ are each equicontinuous on $[0, R]$. We have two sequences, $\{u_k\}$ and $\{v_k\}$, that are uniformly bounded and equicontinuous on $[0, R]$. By the Arzela-Ascoli Theorem (Preliminary 2-1) there exists a $u \in C[0, R]$ and a subsequence $\{u_{k_j}\}$ such that

$$u_{k_j} \rightarrow u \text{ uniformly on } [0, R].$$

If we consider the corresponding subsequence $\{v_{k_j}\} \subset \{v_k\}$, the Arzela-Ascoli Theorem again implies there exists a $v \in C[0, R]$ and a subsequence of $\{v_{k_j}\}$, call it $\{v_{k_{j_i}}\}$, such that

$$v_{k_{j_i}} \rightarrow v \text{ uniformly on } [0, R].$$

Therefore we have

$$(u_{k_{j_i}}, v_{k_{j_i}}) \rightarrow (u, v) \text{ uniformly on } [0, R] \times [0, R].$$

Hence, (u, v) is a fixed point of (4.8) in $C[0, R] \times C[0, R]$. Next, we extend this result to show T has a fixed point in $C[0, \infty) \times C[0, \infty)$. We proceed with a diagonal argument similar to that in Theorem 3-2. However, in this case, we are considering fixed points and convergence in $C[0, \infty) \times C[0, \infty)$ rather than $C[0, \infty)$. The idea is the same.

Let $\{(w_k, z_k)\}$ be a sequence of fixed points defined by

$$T(w_k, z_k) = (w_k, z_k) \text{ on } [0, k], (w_k, z_k) \in C[0, k] \times C[0, k]. \quad (4.14)$$

for $k = 1, 2, \dots$. As earlier, we may show that both $\{w_k\}$ and $\{z_k\}$ are bounded and equicontinuous on $[0, 1]$. Thus by applying the Arzela-Ascoli Theorem to each sequence separately, we can derive $\{(w_k, z_k)\}$ contains a convergent subsequence, $\{(w_k^1, z_k^1)\}$, that converges uniformly on $[0, 1] \times [0, 1]$. Let

$$(w_k^1, z_k^1) \rightarrow (u_1, v_1) \text{ uniformly on } [0, 1] \times [0, 1] \text{ as } k \rightarrow \infty.$$

Likewise, the subsequences $\{w_k^1\}$ and $\{z_k^1\}$ are each bounded and equicontinuous on $[0, 2]$ so there exists a subsequence $\{(w_k^2, z_k^2)\}$ of $\{(w_k^1, z_k^1)\}$ such that

$$(w_k^2, z_k^2) \rightarrow (u_2, v_2) \text{ uniformly on } [0, 2] \times [0, 2] \text{ as } k \rightarrow \infty.$$

Notice that $\{(w_k^2, z_k^2)\} \subseteq \{(w_k^1, z_k^1)\} \subseteq \{(w_k^1, z_k^1)\}_{k=2}^\infty$ so $(u_2, v_2) = (u_1, v_1)$ on $[0, 1] \times [0, 1]$.

Continuing, we obtain a sequence $\{(u_k, v_k)\}$ such that

$$\begin{aligned} (u_k, v_k) &\in C[0, k] \times C[0, k] & k = 1, 2, \dots \\ (u_k(r), v_k(r)) &= (u_1(r), v_1(r)) & \text{for } r \in [0, 1] \\ (u_k(r), v_k(r)) &= (u_2(r), v_2(r)) & \text{for } r \in [0, 2] \\ &\vdots \\ (u_k(r), v_k(r)) &= (u_{k-1}(r), v_{k-1}(r)) & \text{for } r \in [0, k-1] \end{aligned}$$

Thus (u_k, v_k) converges to (u, v) which satisfies

$$(u(r), v(r)) = (u_k(r), v_k(r)) \text{ if } 0 \leq r \leq k. \quad (4.15)$$

The convergence is uniform on bounded sets, and thus $(u(r), v(r)) \in C[0, \infty) \times C[0, \infty)$ is a fixed point of (4.8) and an entire solution to (1.2). We chose our central values $0 < a \leq \bar{u}(0)$ and $0 < b \leq \bar{u}(0)$ arbitrarily where \bar{u} is defined in (4.10). Therefore $[0, \bar{u}(0)] \times [0, \bar{u}(0)]$ is a subset of our set of central values S given in (4.1). We conclude (1.2) has an infinitely many entire solutions. \square

Note we did not use the function G from in (4.3) directly in this result. Rather, since G satisfied the Keller-Osserman condition (2.2), we were guaranteed H satisfied this condition according to Lemma 4-1. We then used H in our argument. The function G need not satisfy (2.2) to show existence of entire solutions. In fact, we only need f or g to satisfy the Keller-Osserman condition, which we prove in the following corollary.

Corollary 4-1 Assume $p, q \in C(\mathbb{R}^n)$, are nonnegative, not identically zero at infinity, and satisfy (2.10). Let $f, g \in C[0, \infty)$ satisfy (2.12). Also, suppose f or g satisfies the Keller-Osserman condition (2.2). Then system (1.2) has infinitely many entire nonnegative solutions.

Proof. Without loss of generality, suppose f satisfies (2.2). Then H as defined in (4.3) still satisfies the Keller-Osserman condition because

$$\int_1^\infty \left[\int_0^t H(s) ds \right]^{-1/2} dt \leq \int_1^\infty \left[\int_0^t f(s) ds \right]^{-1/2} dt < \infty.$$

In addition, H clearly must still satisfy (2.12). Our argument then proceeds exactly like the proof of Theorem 4-1. □

While we may have the existence of entire solutions to system (1.2) when G does not satisfy (2.2), it is necessary for G to satisfy the Keller-Osserman condition as we characterize our set of central values. We examine this set next.

Theorem 4-2 Given the hypotheses in Theorem 4-1, the set of central values S , given in (4.1), is closed and bounded.

Proof. To show S is bounded, we proceed in a manner comparable to Cirstea and Radulescu [6]. For contradiction, suppose S is unbounded. Consider the equation

$$\begin{aligned}\Delta\eta &= \psi(|x|)G(\eta/2), \quad x \in \mathbb{R}^n, \\ \eta &\rightarrow \infty \text{ as } |x| \rightarrow \infty\end{aligned}\tag{4.16}$$

where ψ and G are given in (4.2) and (4.3). Recall ψ is nonnegative, c-positive, and satisfies (2.10). Also, G satisfies (2.2), so an entire large solution η exists for equation (4.16) by Theorem 2 in [12]. Since we are assuming S is unbounded, we can find central values $a, b \in S$ such that $a + b > \eta(0)$. Notice that

$$\begin{aligned}f(v(r)) &\geq G(v(r)) \geq G\left(\frac{u(r)+v(r)}{2}\right) \text{ if } v(r) \geq u(r), \\ g(u(r)) &\geq G(u(r)) \geq G\left(\frac{u(r)+v(r)}{2}\right) \text{ if } u(r) \geq v(r).\end{aligned}\tag{4.17}$$

Then using (4.17), we have

$$\begin{aligned}\Delta(u+v) &= p(r)f(v) + q(r)g(u) \\ &\geq \psi(r)(f(v) + g(u)) \\ &\geq \psi(r)G\left(\frac{u(r)+v(r)}{2}\right).\end{aligned}\tag{4.18}$$

Consider some closed finite ball $B(0, R)$. We will use the maximum principle to show $u + v \leq \eta$ in this ball.

Since η and $u + v$ are radial, consider these equations in their related ordinary differential equation form. Define $h(r) = (1 + r^2)^{-1/2}$, and suppose for contradiction that $u + v > \eta$ at some point in $[0, R]$. Let $\varepsilon > 0$ be small enough such that

$$\max_{r \in [0, R]} [(u+v)(r) - \eta(r) - \varepsilon h(r)] > 0.$$

Let $r_0 \in [0, R]$ be the point where this maximum occurs so $(u+v)(r_0) - \eta(r_0) - \varepsilon h(r_0) > 0$.

Therefore at r_0 , by (4.18) and our assumption that $u(r_0) + v(r_0) > \eta(r_0)$, we have

$$\begin{aligned}
0 &\geq \Delta((u+v)(r_0) - \eta(r_0) - \varepsilon h(r_0)) \\
&\geq \psi(r_0)G\left(\frac{u(r_0) + v(r_0)}{2}\right) - \psi(r_0)G(\eta/2) - \Delta\varepsilon h(r_0) \\
&= \psi(r_0)\left[G\left(\frac{u(r_0) + v(r_0)}{2}\right) - G(\eta/2)\right] - \Delta\varepsilon h(r_0) \\
&\geq -\varepsilon\Delta h(r_0) \\
&> 0,
\end{aligned} \tag{4.19}$$

where the last inequality holds because $\Delta h(r) < 0$ for all $r \in [0, R]$. This may be seen by direct calculation, and is also a specific form of Lemma 2.0.18 of Proano [23]. We have a contradiction, and thus

$$u + v \leq \eta \text{ in } B(0, R). \tag{4.20}$$

However, this now contradicts our original assumption that a, b were chosen such that

$$u(0) + v(0) = a + b > \eta(0).$$

Therefore, we conclude that S is bounded.

Now we prove S is closed by showing S contains its boundary. Let $(a_0, b_0) \in \partial S$. Then there exists some ball centered at (a_0, b_0) with radius $1/k$ such that $B((a_0, b_0), 1/k) \cap S \neq \emptyset$.

For each $k \geq 1$, we denote the arbitrary point $(a_0^k, b_0^k) \in S \cap B((a_0, b_0), 1/k)$. Note

$\{(a_0^k, b_0^k)\} \rightarrow (a_0, b_0)$ as $k \rightarrow \infty$. Then, we define the sequence

$$(u_k(r), v_k(r)) = \begin{cases} a_0^k + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) f(v_k(s)) ds dt, & r \geq 0, \\ b_0^k + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) g(u_k(s)) ds dt, & r \leq 0 \end{cases} \quad (4.21)$$

where each (u_k, v_k) is an entire solution to (1.2). These solutions exist since each central value $(a_0^k, b_0^k) \in B((a_0, b_0), 1/k) \cap S \subset S$. We now show $\{(u_k, v_k)\}$ has a convergent subsequence on $C[0, \infty) \times C[0, \infty)$. Similar to our argument for Theorem 4-1, we first demonstrate that the sequence has a convergent subsequence on $C[0, R] \times C[0, R]$ for arbitrary R , and then we extend to $C[0, \infty) \times C[0, \infty)$. We must make several minor adjustments to achieve our bounds for this new sequence, but the idea is comparable.

For each $k = 1, 2, \dots$, (4.20) gives us $u_k + v_k \leq \eta$ for $r \in [0, R]$ where η is defined in (4.16). Since $\eta'(r) \geq 0$, we have

$$u_k(r) + v_k(r) \leq \eta(R) < \infty \text{ for } 0 \leq r \leq R \quad (4.22)$$

where $\eta(R)$ is finite because η is entire. Thus $\{u_k\}, \{v_k\}$ are each uniformly bounded on $[0, R]$.

Lemma 2-1 implies $\{u_k\}, \{v_k\}$ are equicontinuous on $[0, R]$. As in the previous proof, we may

use the Arzela-Ascoli Theorem to show a convergent subsequence of (4.21) exists on

$C[0, R] \times C[0, R]$. This gives us a solution to system (1.2) in $B(0, R) \subset \mathbb{R}^n$ with central values

$(a_0, b_0) \in \partial S$. Since R is arbitrary, we can use the same diagonal argument from Theorem 4-1 to

show (4.21) has a convergent subsequence on $C[0, \infty) \times C[0, \infty)$. Therefore we have the solution

$$\begin{cases} u(r) = a_0 + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) f(v(s)) ds dt, & r \geq 0, \\ v(r) = b_0 + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) g(u(s)) ds dt, & r \geq 0 \end{cases}$$

where $(a_0, b_0) \in \partial S$. Hence $(a_0, b_0) \in S$ implying S is closed. This completes our proof. \square

Thus far, we have established the existence of entire solutions to system (1.2) and characterized the set of central values S as closed and bounded. For monotonic and convex functions f and g , Peng and Song [22] further characterized S as convex. We cannot show this for our non-monotone problem. In fact, we will provide an example later for which S (found numerically) appears non-convex. Still, Theorem 4-1 and Theorem 4-2 provide a limited geometric description of S which we present as the following corollary.

Corollary 4-2 Given the hypotheses in Theorem 4-1, our set of central values S given in (4.1), satisfies $T_1 \subseteq S \subseteq T_2$ where $T_1 = \{(a, b) : 0 \leq a \leq \bar{u}(0), 0 \leq b \leq \bar{u}(0)\}$ and $T_2 = \{(a, b) : a + b \leq \eta(0)\}$. The functions \bar{u} and η are given in (4.10), and (4.16) respectively.

See Figure 4-1.

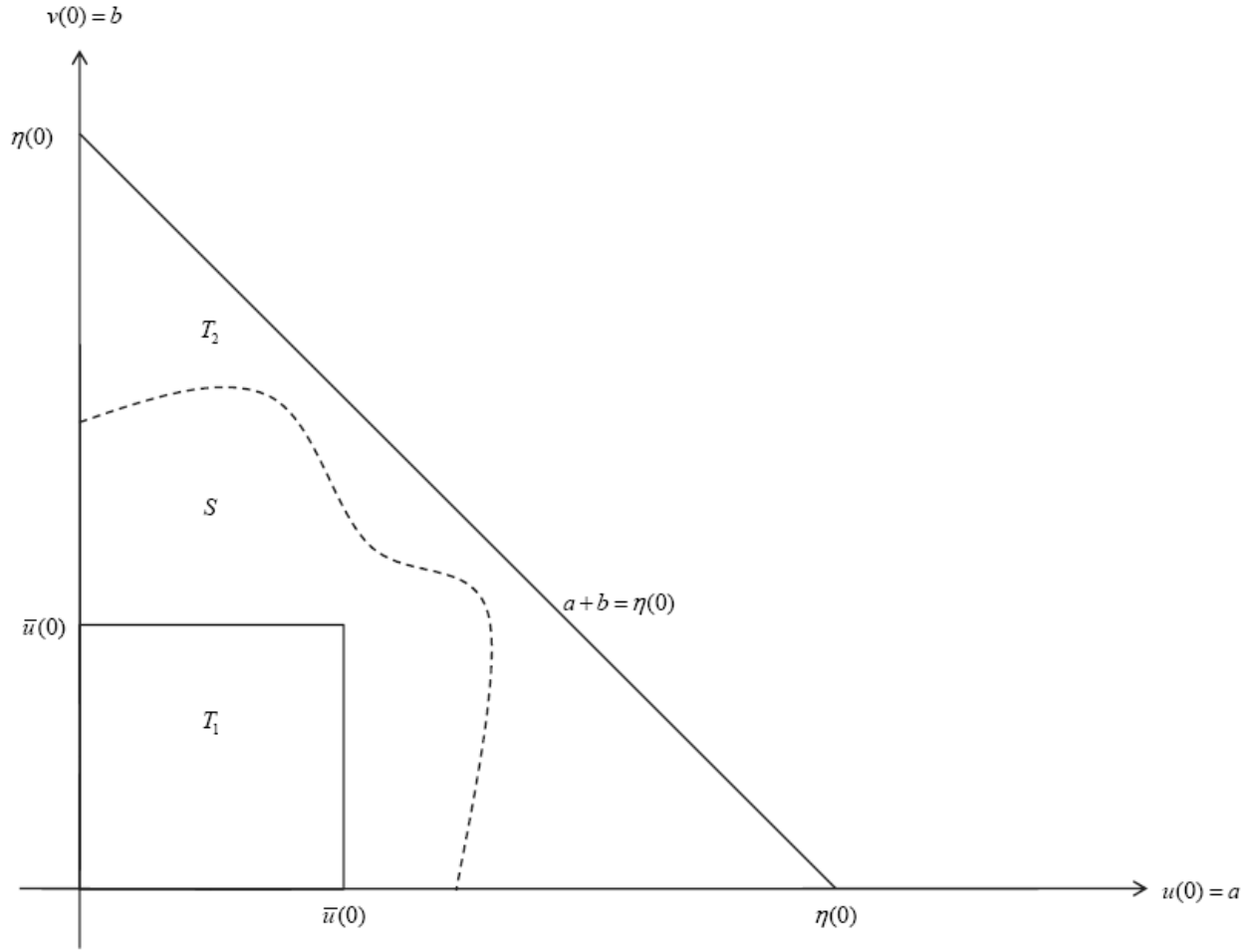


Figure 4-1. Entire Solution Existence Region for Central Values of (1.2)

Proof. In Theorem 4-1, we proved the existence of entire solutions to (1.2) by constructing $T_1 \subseteq S$. In Theorem 4-2, we demonstrated S to be bounded by constructing $T_2 \supseteq S$. \square

Note that we drew S in Figure 4-1 as connected, but we have not shown this to be the case. Indeed, we know much less about our set of central values as compared to monotonic problems. However, the fact that S is closed and bounded is enough to show the existence of entire large solutions. Before we proceed, we will need to prove several minor lemmas.

Lemma 4-3 Under the hypotheses of Theorem 4-1 and for any $(c, d) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $(c, d) \notin S$ and $c \neq 0 \neq d$, system (1.2) has a solution in some ball $B(0, \rho)$, where $0 < \rho < \infty$.

Proof. Let (c, d) be defined as above. We wish to find a radial solution to (1.2) in $B(0, \rho)$, $\rho > 0$. This solution is a fixed point of the operator $T : C[0, \rho] \times C[0, \rho] \rightarrow C[0, \rho] \times C[0, \rho]$ defined by

$$T(u(r), v(r)) \equiv (\hat{u}(r), \hat{v}(r)) \equiv \begin{cases} c + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) f(v(s)) ds dt, & 0 \leq r \leq \rho, \\ d + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) g(u(s)) ds dt, & 0 \leq r \leq \rho. \end{cases} \quad (4.23)$$

We will show for ρ sufficiently small, a fixed point, and thereby a solution exists. We accomplish this using the version of Schauder's Fixed Point Theorem given in Lemma 4-2. First, we establish a subset $X \subset C[0, \rho] \times C[0, \rho]$ that satisfies the necessary hypotheses of this lemma. Note that for $\rho > 0$, $C[0, \rho] \times C[0, \rho]$ is a Banach space with norm

$\|(u, v)\|_\infty = \max(\|u\|_\infty, \|v\|_\infty)$ where $\|u\|_\infty = \sup_{0 \leq r \leq \rho} |u(r)|$. Define the subset $X \subset C[0, \rho] \times C[0, \rho]$ by

$$X = \{(u, v) \in C[0, \rho] \times C[0, \rho] : \|(u, v) - (c, d)\|_\infty \leq \min\{c, d\}\},$$

where c, d are our given central values. Since this is a closed ball in a Banach space, X is closed, bounded, and convex

We turn our attention to the operator T given in (4.23). We show T is a compact operator. Let Y be an arbitrary bounded set given by

$$Y = \{(u, v) \in C[0, \rho] \times C[0, \rho] : \|(u, v)\|_\infty \leq M\}$$

for some $M > 0$. Then, suppose (u_k, v_k) is an arbitrary sequence in Y , and consider the image of this sequence $T(u_k, v_k) = (\hat{u}_k, \hat{v}_k)$. Clearly

$$\begin{aligned} \hat{u}_k(r) &= c + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) f(v_k(s)) ds dt \\ &\leq c + \int_0^r t^{1-n} \int_0^t s^{n-1} \phi(s) H(v_k(s)) ds dt \\ &\leq c + \int_0^\rho t^{1-n} \int_0^t s^{n-1} \phi(s) H(M) ds dt \\ &= M_0 \end{aligned}$$

where ϕ and H are given in (4.2) and (4.3), and M_0 is constant. Similarly $\hat{v}_k(r) \leq M_0$. Thus, \hat{u}_k and \hat{v}_k are uniformly bounded. Further, since $u_k(r) < M$ and $v_k(r) < M$ for all k and all $0 \leq r \leq \rho$, Lemma 2-1 implies that \hat{u}_k and \hat{v}_k are also equicontinuous on $0 \leq r \leq \rho$. Finally, using the Arzela-Ascoli Theorem as before, there exists a uniformly convergent subsequence $(\hat{u}_{k_j}, \hat{v}_{k_j}) \rightarrow (\hat{u}, \hat{v})$ where $(\hat{u}, \hat{v}) \in C[0, \rho] \times C[0, \rho]$. This implies that the image of an arbitrary bounded set under T , is (sequentially) compact. That is, T is a compact operator.

Lastly, we must show that T maps elements of X back into X . Again, take any $(u, v) \in X$. Since X is bounded, suppose $\|(u, v)\|_\infty \leq Q$. By integrating as in (3.8) and (3.10), we have the estimate

$$\begin{aligned}
\int_0^r t^{1-n} \int_0^t s^{n-1} p(s) f(v(s)) ds dt &\leq \int_0^r t^{1-n} \int_0^t s^{n-1} \phi(s) H(v(s)) ds dt \\
&\leq H(v(r)) \int_0^r t^{1-n} \int_0^t s^{n-1} \phi(s) ds dt \\
&\leq H(Q) \int_0^r t \phi(t) dt \\
&\leq H(Q) \rho \int_0^\rho \phi(t) dt
\end{aligned}$$

where ϕ and H are given in (4.2) and (4.3). Similarly, we may show

$$\int_0^r t^{1-n} \int_0^t s^{n-1} q(s) g(u(s)) ds dt \leq H(Q) \rho \int_0^\rho \phi(t) dt .$$

Since $\phi(r)$ is well defined for $r \in [0, \infty)$ and $c, d > 0$ by hypothesis, we can choose $\rho > 0$ small

enough so that $H(Q) \rho \int_0^\rho \phi(t) dt < \min\{c, d\}$. Doing so, for $(u, v) \in X$, we have

$T(u(r), v(r)) = (\hat{u}(r), \hat{v}(r))$ where

$$\begin{aligned}
c &\leq \hat{u}(r) \\
&= c + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) f(v(s)) ds dt \\
&\leq c + H(Q) \rho \int_0^\rho \phi(t) dt \\
&\leq c + \min\{c, d\},
\end{aligned}$$

and similarly $d \leq \hat{v}(r) \leq d + \min\{c, d\}$ for all $0 \leq r \leq \rho$. Thus $(\hat{u}, \hat{v}) \in X$.

We have shown X to be closed, convex, and bounded. Also, we have proven the operator T in (4.23) is compact, and we have shown for any $x \in X$, $Tx \in X$. It follows from

Lemma 4-2 that (4.23) has a fixed point. Hence system (1.2) has a solution in $B(0, \rho)$. Further, we chose $(c, d) \notin S$ implying $\rho < \infty$. \square

Now that we have established the existence of solutions for any pair of positive central values, we will show for any such $(c, d) \notin S$, a large solution exists on a finite domain.

Lemma 4-4 Given the hypotheses of Theorem 4-1, let (u, v) be a solution to (1.2) with central values $(c, d) \notin S$, $c \neq 0 \neq d$, and define the set

$$R_{sol} = \{r > 0 : \text{there exists a solution of (1.2) in } B(0, r) \text{ such that } (u(0), v(0)) = (c, d)\}. \quad (4.24)$$

Let $R_{c,d}$ be given as

$$R_{c,d} = \sup R_{sol}. \quad (4.25)$$

Then $\lim_{r \rightarrow R_{c,d}} u(r) = \infty = \lim_{r \rightarrow R_{c,d}} v(r)$.

Proof. Take $(c, d) \notin S$, $c \neq 0 \neq d$. By Lemma 4-3, $R_{sol} \neq \emptyset$, and since $(c, d) \notin S$, $R_{c,d} < \infty$.

Let (u, v) be a solution of (1.2) in $B(0, R_{c,d})$ with central values (c, d) . Since $u' \geq 0$ and $v' \geq 0$,

$\lim_{r \rightarrow R_{c,d}} u(r)$ and $\lim_{r \rightarrow R_{c,d}} v(r)$ exist (possibly infinity). For contradiction, suppose $\lim_{r \rightarrow R_{c,d}} u(r) = A < \infty$.

This implies

$$\begin{aligned}
\lim_{r \rightarrow R_{c,d}} v(r) &= \lim_{r \rightarrow R_{c,d}} d + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) g(u(s)) ds dt \\
&\leq d + \int_0^{R_{c,d}} t^{1-n} \int_0^t s^{n-1} q(s) H(u(s)) ds dt \\
&\leq d + H(A) \int_0^{R_{c,d}} t^{1-n} \int_0^t s^{n-1} q(s) ds dt \\
&= B < \infty.
\end{aligned}$$

Thus $u(R_{c,d}) = A$ and $v(R_{c,d}) = B$ are well defined. We now consider system (1.2) over the interval $R_{c,d} \leq r \leq R_{c,d} + \varepsilon$, where $\varepsilon > 0$. Again, our equations are radial, so we may integrate system (1.2) to obtain

$$\begin{cases} u(r) = A + \int_{R_{c,d}}^r t^{1-n} \int_0^t s^{n-1} p(s) f(v(s)) ds dt, & R_{c,d} \leq r \leq R_{c,d} + \varepsilon, \\ v(r) = B + \int_{R_{c,d}}^r t^{1-n} \int_0^t s^{n-1} q(s) g(u(s)) ds dt, & R_{c,d} \leq r \leq R_{c,d} + \varepsilon. \end{cases} \quad (4.26)$$

This is equivalent to the problem

$$\begin{cases} u'(r) = r^{1-n} \int_0^r s^{n-1} p(s) f(v(s)) ds, & u(R_{c,d}) = A, & R_{c,d} \leq r \leq R_{c,d} + \varepsilon, \\ v(r) = B + \int_{R_{c,d}}^r t^{1-n} \int_0^t s^{n-1} q(s) g(u(s)) ds dt, & & R_{c,d} \leq r \leq R_{c,d} + \varepsilon. \end{cases} \quad (4.27)$$

Substituting $v(r)$ into our equation for $u'(r)$, we get the initial value problem

$$\begin{aligned}
u'(r) &= r^{1-n} \int_0^r s^{n-1} p(s) f(v(s)) ds \\
&= r^{1-n} \int_{R_{c,d}}^r s^{n-1} p(s) f \left[B + \int_{R_{c,d}}^s t^{1-n} \int_0^t y^{n-1} q(y) g(u(y)) dy dt \right] ds \\
&\equiv F(r, u(r))
\end{aligned} \tag{4.28}$$

where $R_{c,d} \leq r \leq R_{c,d} + \varepsilon$ and $u(R_{c,d}) = A$. If we show this initial value problem has a solution on some interval $R_{c,d} \leq r \leq R_{c,d} + \beta$ ($0 < \beta \leq \varepsilon$), then (4.27) and (4.26) will have a solution as well. This would imply that system (1.2) has a solution on $0 \leq r \leq R_{c,d} + \beta$, contradicting the definition of $R_{c,d}$ in (4.25). We use Caratheodory's Theorem shown in Preliminary 4-3.

Define the rectangle

$$R = \{(r, u) : |r - R_{c,d}| \leq \varepsilon, |u - A| \leq A\} \tag{4.29}$$

where $\varepsilon > 0$, $A = \lim_{r \rightarrow R_{c,d}} u(r)$, and $R_{c,d}$ is given in (4.25). Treating r and u as two independent

variables, we see

$$\begin{aligned}
F(r, u) &= r^{1-n} \int_{R_{c,d}}^r s^{n-1} p(s) f \left[B + \int_{R_{c,d}}^s t^{1-n} \int_0^t y^{n-1} q(y) g(u) dy dt \right] ds \\
&= r^{1-n} \int_{R_{c,d}}^r s^{n-1} p(s) f \left[B + g(u) \int_{R_{c,d}}^s t^{1-n} \int_0^t y^{n-1} q(y) dy dt \right] ds.
\end{aligned} \tag{4.30}$$

Recall f, g, p, q are all defined on $[0, \infty)$, and consider $F(r, u)$ on $[R_{c,d} - \rho, R_{c,d} + \rho] \times [0, 2A]$.

Let $N = \max_{0 \leq u \leq 2A} g(u)$, and choose ρ small enough so that

$$\begin{aligned}
g(u) \int_{R_{c,d}}^{R_{c,d}-\rho} t^{1-n} \int_0^t y^{n-1} q(y) dy dt &= -g(u) \int_{R_{c,d}-\rho}^{R_{c,d}} t^{1-n} \int_0^t y^{n-1} q(y) dy dt \\
&\geq -N \int_{R_{c,d}-\rho}^{R_{c,d}} t^{1-n} \int_0^t y^{n-1} q(y) dy dt \\
&\geq -B.
\end{aligned}$$

This ensures the argument of f remains positive, and hence $F(r, u)$ is defined on all

$[R_{c,d} - \rho, R_{c,d} + \rho] \times [0, 2A]$. Then, choosing $\varepsilon < \rho$, (4.29) implies

$R \subset [R_{c,d} - \rho, R_{c,d} + \rho] \times [0, 2A]$. Also, since R is closed and bounded, $\max_{(r,u) \in R} F(r, u) = M$ is well

defined, and we have

$$|F(r, u)| = F(r, u) \leq M \text{ for all } (r, u) \in R. \quad (4.31)$$

Applying Cartheodory's Theorem (Preliminary 4-3), (4.28) has a solution in

$R_{c,d} - \beta \leq r \leq R_{c,d} + \beta$ ($0 < \beta \leq \varepsilon$) with $u(R_{c,d}) = A$. We have our contradiction, and therefore

$\lim_{r \rightarrow R_{c,d}} u(r) = \infty$. Similarly, we can obtain $\lim_{r \rightarrow R_{c,d}} v(r) = \infty$, and our proof is complete. \square

We finally have the necessary tools to present our results for existence of entire large solutions for systems. Our general argument is taken from Lair and Wood [13].

Theorem 4-3 Given the set of central values S in (4.1), let

$$E \equiv \{(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+ : (a, b) \in \partial S, a \neq 0, b \neq 0\}. \quad (4.32)$$

Under the hypotheses of Theorem 4-1, any positive radial solution of system (1.2) with central values $(a, b) \in \bar{E}$ is an entire large solution.

Proof. Let $(a, b) \in E$. We have shown in Theorem 4-2 that $\partial S \subset S$, so define (u, v) to be an entire solution with central values (a, b) . Since $(a, b) \in \partial S$, we may choose a point $(a_k, b_k) \notin S$, $a_k > 0, b_k > 0$, where $(a_k, b_k) \in B((a, b), 1/k)$. From Lemma 4-3, there exists a solution to (1.2) in the finite ball $B(0, R_{a_k, b_k})$ with central value (a_k, b_k) . For simplicity, we will abbreviate $R_{a_k, b_k} \equiv R_k$. We write each solution as

$$\begin{cases} u_k(r) = a_k + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) f(v_k(s)) ds dt, & r \in [0, R_k), \\ v_k(r) = b_k + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) g(u_k(s)) ds dt, & r \in [0, R_k). \end{cases}$$

From Lemma 4-4, we have for each R_k ,

$$\lim_{r \rightarrow R_k} u_k(r) = \lim_{r \rightarrow R_k} v_k(r) = \infty.$$

Note that if $\{R_k\}$ is bounded, then there exists a convergent subsequence $R_{k_j} \rightarrow R$. We will show $R = \infty$, implying $\{R_k\}$ is unbounded. In doing so, we will prove the existence of entire large solutions. Notice that since v_k , u_k , and H from (4.3) are non-decreasing, we have

$$\begin{aligned}
v_k(r) &= b_k + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) g(u_k(s)) ds dt \\
&\leq b_k + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) H(u_k(s)) ds dt \\
&\leq b_k + H(u_k(r)) \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) ds dt \\
&\leq \frac{b_k}{a_k} \left(a_k + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) f(v_k(s)) ds dt \right) + H(u_k(r)) \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) ds dt \\
&\leq C_1 u_k(r) + C_2 H(u_k(r)).
\end{aligned} \tag{4.33}$$

The value C_1 is any bound on b_k / a_k , which exists since $a \neq 0$, and

$$C_2 = \int_0^\infty t^{1-n} \int_0^t s^{n-1} q(s) ds dt \leq \frac{1}{n-2} \int_0^\infty s q(s) ds < \infty.$$

This inequality for C_2 is obtained by integrating as in (3.8) and (3.10) and is finite from (2.10).

We then define $h(t) = H(C_1 t + C_2 H(t))$. Using the results of Lemma 4-1, we have

$$\begin{aligned}
h(0) &= H(0 + C_2 H(0)) = H(0) = 0, \\
h(s) &= H(C_1 s + C_2 H(s)) > 0, \quad s > 0,
\end{aligned} \tag{4.34}$$

and, choosing $C_1 \geq 1$,

$$\begin{aligned}
\int_1^\infty \left[\int_0^t h(s) ds \right]^{-1/2} dt &= \int_1^\infty \left[\int_0^t H(C_1 s + C_2 H(s)) \right]^{-1/2} dt \\
&\leq \int_1^\infty \left[\int_0^t H(s) \right]^{-1/2} dt \\
&< \infty.
\end{aligned} \tag{4.35}$$

That is, h satisfies (2.12) and the Keller-Osserman condition (2.2). Lemma 1 in [12] then guarantees that

$$F(s) = \int_s^\infty \frac{dt}{h(t)} \quad (4.36)$$

is well defined for all $s > 0$. Note that

$$F'(s) = \frac{-1}{h(s)} < 0 \text{ and } F''(s) = \frac{h'(s)}{[h(s)]^2} > 0. \quad (4.37)$$

We have from (4.33) that

$$\begin{aligned} \Delta u_k &= p(r)f(v_k) \\ &\leq p(r)H(v_k) \\ &\leq p(r)H[C_1 u_k(r) + C_2 H(u_k(r))] \\ &= p(r)h(u_k). \end{aligned}$$

Then, using (4.37), we calculate

$$\begin{aligned} \Delta F(u_k) &= F'(u_k)\Delta u_k + F''(u_k)|\nabla u_k|^2 \\ &= \frac{-1}{h(u_k)}\Delta u_k + \frac{h'(u_k)}{[h(u_k)]^2}|\nabla u_k|^2 \\ &\geq \frac{-1}{h(u_k)}p(r)h(u_k) \\ &= -p(r). \end{aligned}$$

Rewriting the Laplacian in radial form and multiplying each side by r^{n-1} , we obtain

$$\left(r^{n-1} \frac{d}{dr} F(u_k) \right)' \geq -r^{n-1} p(r).$$

Integrating over $[0, r]$ where $0 < r < R_k$ gives us

$$\frac{d}{dr} F(u_k) \geq -r^{1-n} \int_0^r s^{n-1} p(s) ds.$$

Next, we integrate over $[r, R_k]$. Notice that since $u_k(r) \rightarrow \infty$ as $r \rightarrow R_k$, we see from (4.36)

$F(u_k(r)) \rightarrow 0$ as $r \rightarrow R_k$. Thus integration yields

$$-F(u_k(r)) \geq - \int_r^{R_k} t^{n-1} \int_0^r s^{n-1} p(s) ds dt.$$

That is,

$$F(u_k(r)) \leq \int_r^{R_k} t^{n-1} \int_0^r s^{n-1} p(s) ds dt.$$

Since $F'(s) < 0$ for $s > 0$, we have

$$u_k(r) \geq F^{-1} \left(\int_r^{R_k} t^{n-1} \int_0^r s^{n-1} p(s) ds dt \right).$$

Now, we let $k \rightarrow \infty$ so $R_k \rightarrow R$ and $u_k \rightarrow u$. We have

$$F^{-1} \left(\int_r^R t^{n-1} \int_0^r s^{n-1} p(s) ds dt \right) \leq u(r).$$

Letting $r \rightarrow R$, and since $\lim_{s \rightarrow \infty} F(s) = 0$ implies $\lim_{s \rightarrow 0^+} F^{-1}(s) = \infty$, we have

$$\lim_{r \rightarrow R} F^{-1} \left(\int_r^R t^{n-1} \int_0^r s^{n-1} p(s) ds dt \right) = \lim_{s \rightarrow 0^+} F^{-1}(s) = \infty \leq \lim_{r \rightarrow R} u(r).$$

However, recall $u(|x|)$ and $v(|x|)$ have central values $(a, b) \in E \subset S$ and are entire. This implies $u(r)$ exists for all $r \in [0, \infty)$. Hence $R = \infty$, and our proof is complete. \square

4.3 Examples

In this section we present several examples of systems that satisfy the restrictions given in our analysis. Systems of the form (1.2) which satisfy the hypotheses of our theorems and lemmas include equations such as

$$\begin{cases} \Delta u = e^{-|x|}(ve^v), \\ \Delta v = e^{-|x|}(ue^u), \end{cases} \quad \begin{cases} \Delta u = 2^{-3|x|}v^2, \\ \Delta v = 4^{-5|x|}u^3, \end{cases} \quad \text{or} \quad \begin{cases} \Delta u = 2^{-|x|}(4e^v - 4), \\ \Delta v = 3^{-|x|}(5e^u - 5). \end{cases} \quad (4.38)$$

In each of these examples, the functions f and g are monotonic. In our results, we also have shown existence of entire large solutions for systems containing non-monotonic functions. For example,

$$\begin{cases} \Delta u = e^{-|x|}(2^v + |4v \sin(v)| - 1), \\ \Delta v = e^{-|x|}(2^u + |4u \sin(u)| - 1), \end{cases} \quad \text{or} \quad \begin{cases} \Delta u = 3^{-|x|}(2^v |\sin v| + (2^v - 1)|\cos v|), \\ \Delta v = 4^{-|x|}(2^u |\sin u| + (2^u - 1)|\cos u|), \end{cases} \quad (4.39)$$

also satisfy the hypotheses of our theorems and lemmas. The most difficult condition to check is that f and g satisfy the Keller-Osserman condition (2.2). We guarantee the examples in (4.39) satisfy this requirement since

$$\begin{aligned} f(s) &= g(s) = 2^s + |4s \sin(s)| - 1 \geq 2^s - 1, \\ f(s) &= g(s) = 2^s |\sin s| + (2^s - 1)|\cos s| \geq 2^s - 1, \end{aligned}$$

and $2^s - 1$ satisfies (2.2). These are only a few examples. If we define our systems piecewise, we can create a wide variety of complicated systems.

Next, we shall consider several of these examples numerically. Consider the simple system

$$\begin{cases} \Delta u = e^{-|x|}(ve^v), \\ \Delta v = e^{-|x|}(ue^u), \end{cases} \quad (4.40)$$

in \mathbb{R}^5 . From our analysis, we know there exists a set of central values S for which entire solutions exist. Also, we know these solutions are large for central values in E given in (4.32). We will numerically solve this radial system by applying a Runge-Kutta algorithm. We first solve (4.40) for central values $(0,0) \in S$. Then, holding $v(0) = 0$ constant, we steadily increase the central value $u(0)$, and numerically solve for each iteration. Figure 4-2 shows these solutions for $u(0)$ ranging from 0 to 2. The dashed lines represents u while the solid lines are solutions v . Figure 4-3 is a closer examination of the transition from entire bounded solutions to large solutions on bounded domains. In other words, Figure 4-3 shows the behavior of solutions as we cross over ∂S .

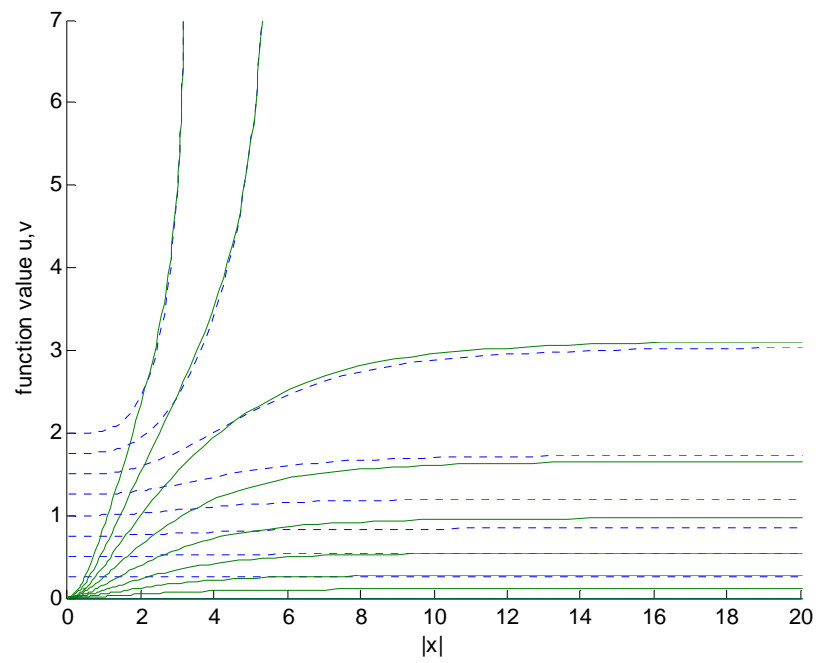


Figure 4-2. Numerical Solutions of (4.40) for Varying Central Values

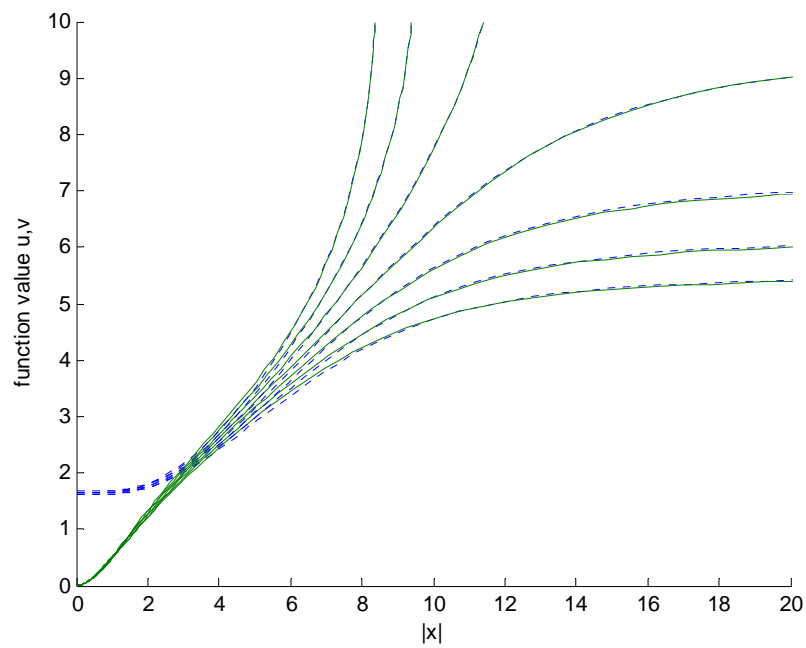


Figure 4-3. Numerical Solutions of (4.40) for Central Values Near Boundary

The solutions for small central values appear bounded and entire. As we increase the central value for u , our solutions increase. Once the central value is large enough, our solutions blow up and exist only on finite domains. This, of course, agrees with the results from our analysis. We can perform similar numerical trials and vary both $u(0)$ and $v(0)$. From this data, we construct the set of central values S for (4.40) as shown in Figure 4-4.

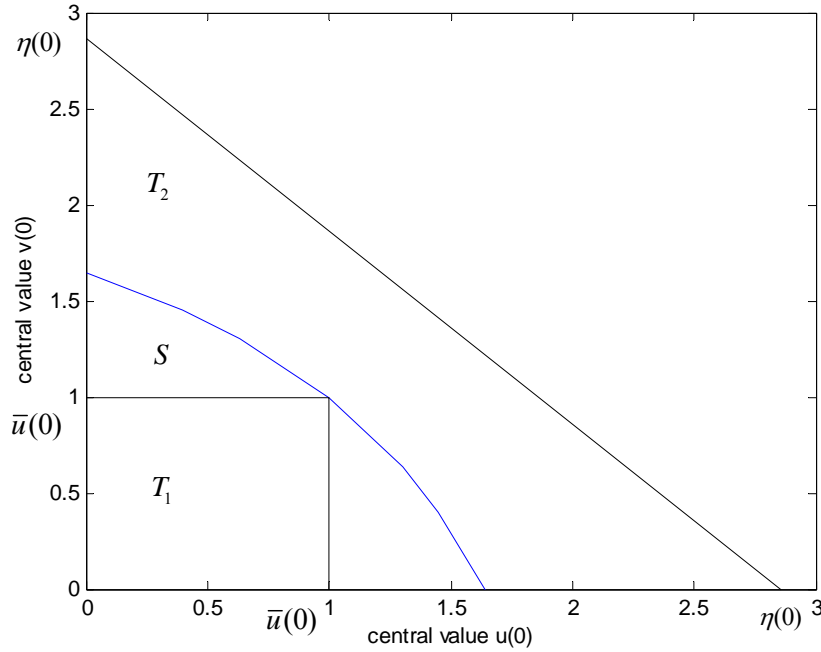


Figure 4-4. Entire Solution Existence Region for Central Values of (4.40)

We have also numerically calculated $\bar{u}(0)$ and $\eta(0)$ as given in (4.10) and (4.16) allowing us to construct T_1 and T_2 from Corollary 4-2. We do this to allow comparison to Figure 4-1. Clearly our numerical results shown in Figure 4-4 agree with our analysis summarized in Figure 4-1.

In the previous example, S had a simple shape due to the monotonicity of our functions and the symmetry of our equations. We will also numerically examine a simple system in which f and g are non-monotonic. Consider the system

$$\begin{cases} \Delta u = e^{-|x|} (2^v + |4v \sin(v)| - 1), \\ \Delta v = 3e^{-|x|} (2^u + |4u \sin(u)| - 1). \end{cases} \quad (4.41)$$

In this case we have $f(s) = g(s) = 2^s + |4s \sin(s)| - 1$ as our non-monotonic functions. Indeed, we plot f in Figure 4-5 to show its non-monotonic behavior.

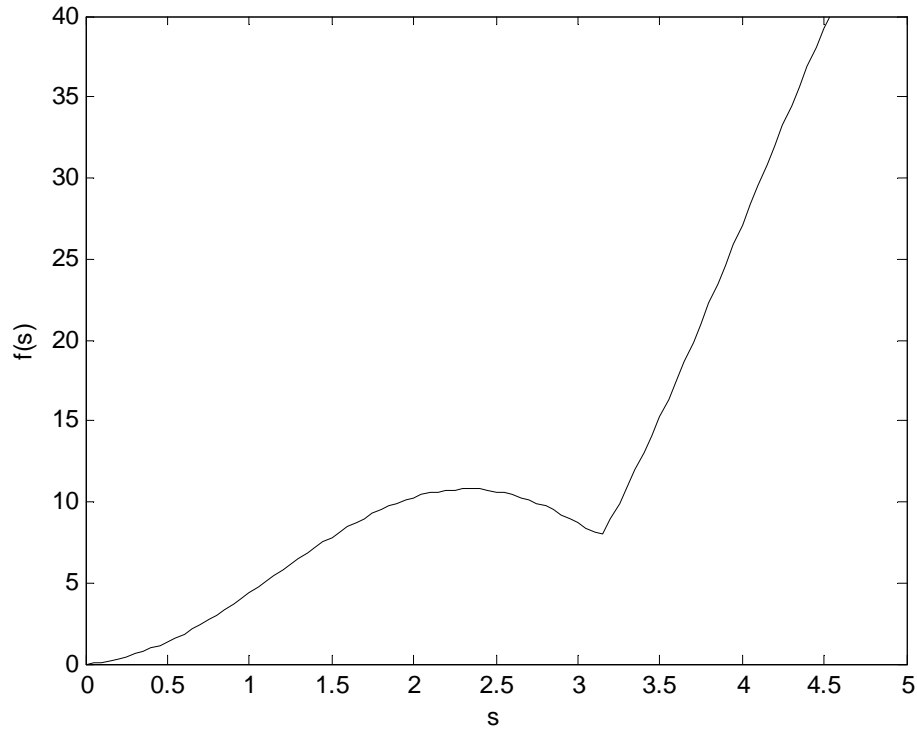


Figure 4-5. Non-monotonic Function f of System (4.41)

Next, we plot numerical solutions for (4.41) with central values near ∂S and $v(0) = 0$ in Figure 4-6.

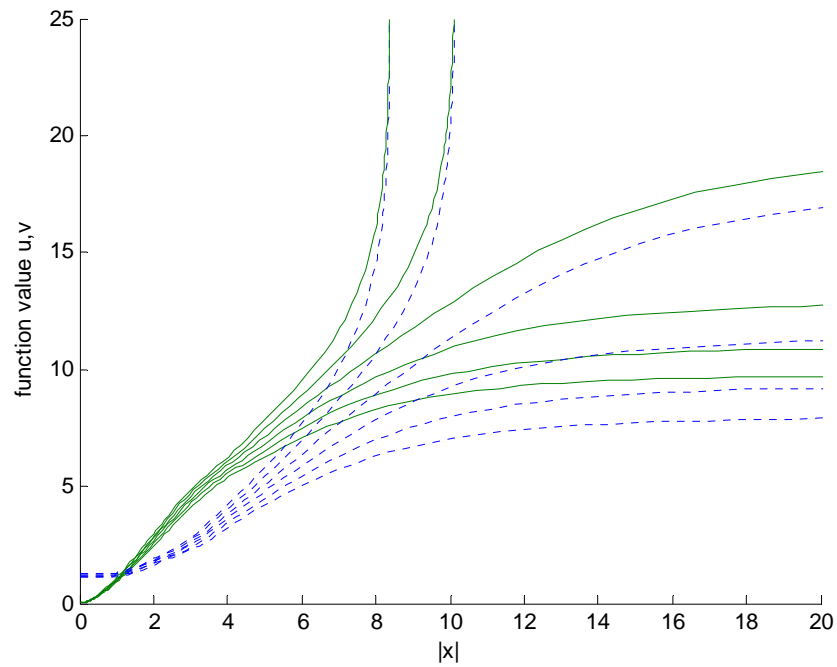


Figure 4-6. Numerical Solutions of (4.41) for Central Values Near Boundary

Our results appear to be similar to those obtained in our first numerical example. However, if we examine these solutions closer, we discover several significant differences. For example, our sequences of functions are not increasing as we increase our central values. Indeed, see Figure 4-7.

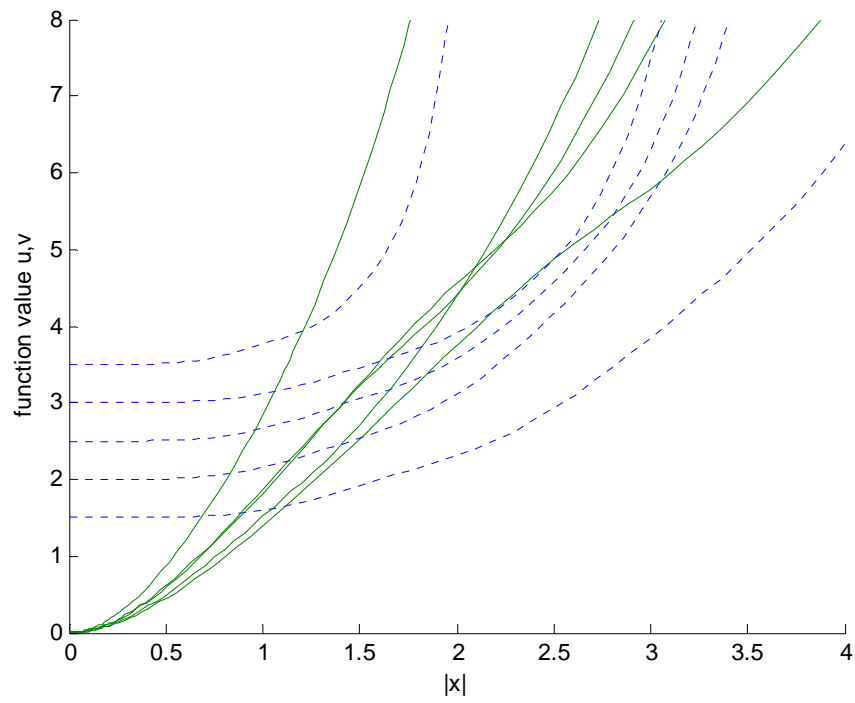


Figure 4-7. Numerical Solutions of (4.41) for Varying Central Values

Also, our set of central values S , shown in Figure 4-8, appears quite different.

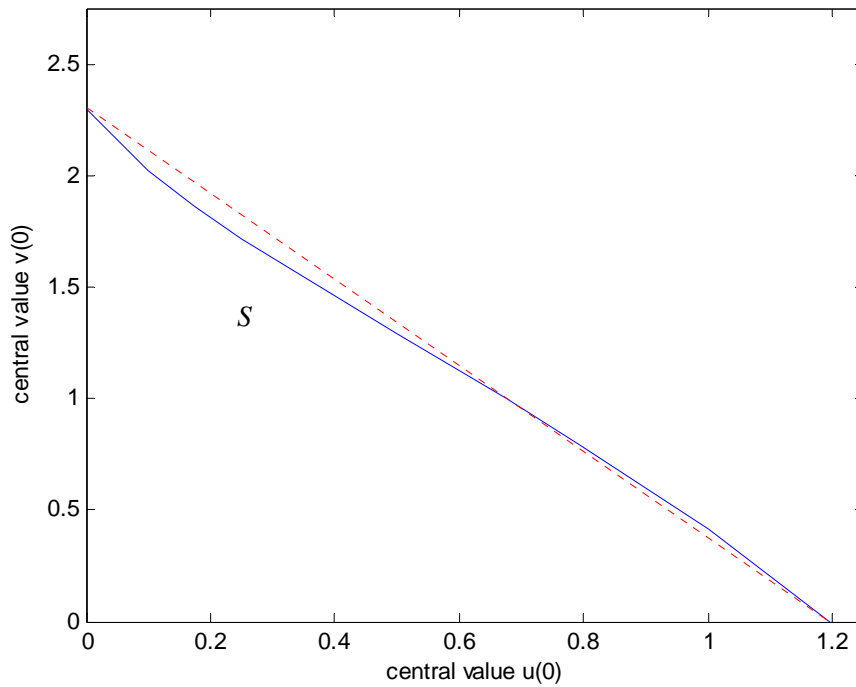


Figure 4-8. Existence Region for Central Values of (4.41)

The dashed line connects the largest central values of the form $(0, v(0)) \in S$ and $(u(0), 0) \in S$.

For this non-monotone example, S does not appear to be convex. If it were, the dashed line should lie entirely in S . Indeed, we have $(0, 2.29670), (1.19673, 0) \in S$, and if S were convex, we can choose $u(0) = 0.2$ and calculate

$$v(0) = \frac{-2.29670}{1.19673}(0.2) + 2.29670 = 1.91286$$

so that $(0.2, 1.91286) \in S$. However, if we solve numerically for these three pairs of central values, we obtain Figure 4-9.

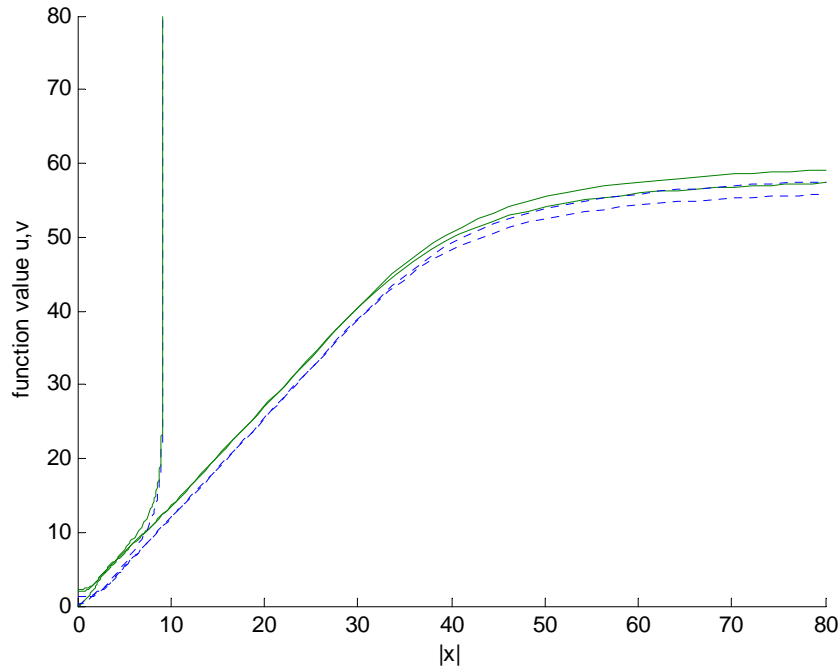


Figure 4-9. Solutions Showing Set of Central Values for (4.41) is Not Convex

Our solution for $(u(0), v(0)) = (0.2, 1.91286)$ is not entire while the remaining central values yield entire bounded solutions. Thus, S appears nonconvex according to our numerical

calculations. This is very interesting since S must be convex for systems with monotone, convex functions f and g (see [13] and [22]).

V. Conclusion

5.1 Summary

Our research began by examining a few open cases left from the work of Smith [26]. We examined semilinear elliptic equations of the form

$$\Delta u = p(x)u^\alpha + q(x)u^\beta, \quad x \in \Omega \subseteq R^n \quad (5.1)$$

for the sublinear case $0 < \alpha \leq \beta \leq 1$. We proved in Theorem 3-1 that (5.1) has entire bounded solutions when nonnegative functions p and q are locally Hölder continuous and (3.1) holds.

We also showed in Theorem 3-2 that (3.14) is necessary and sufficient for (5.1) to have an entire large radial solution. These proofs are extensions of Theorem 22 and Theorem 23 in [26].

Next, we presented several examples of equations which satisfied the hypotheses of our proofs. We found numerical solutions to a few of these problems, and visually demonstrated results of our analysis.

After examining single equations, we considered semilinear elliptic systems. We studied the radial problem

$$\begin{cases} \Delta u = p(|x|)f(v), \\ \Delta v = q(|x|)g(u), \end{cases} \quad x \in \Omega \subseteq R^n. \quad (5.2)$$

Lair and Wood [13] considered (5.2) for $f(v) = v^\alpha$ and $g(u) = u^\beta$. Cirstea and Radulescu [6] and Peng and Song [22] generalized their results by considering f and g non-decreasing. We built on the foundations of these works to establish the existence of entire large positive solutions

of (5.2) for non-monotonic f and g . We only required that G , given in (4.3), satisfy the Keller-Osserman condition (2.2). In our analysis, we established several other notable facts.

First, we demonstrated in Theorem 4-1 that system (5.2) has an infinite number of entire nonnegative solutions. In a corollary, we showed that entire solutions for this system exist even when G fails to satisfy (2.2). We only need that f or g satisfy the Keller-Osserman condition.

Once we established existence of entire solutions, we characterized the set of central values, S . We proved in Theorem 4-2 that this set is closed and bounded, and in a corollary, we provided a rough geometric description of the set. We then considered solutions with central values outside S .

Lemma 4-3 showed for any central values $(a, b) \notin S$, $a > 0, b > 0$, a solution to system (5.2) exists on a bounded domain. This domain is dependent on the central value chosen. Then in Lemma 4-4, we proved that on the largest domain for which these solutions exist, the solution is large. That is $\lim_{r \rightarrow R_{c,d}} u(r) = \infty = \lim_{r \rightarrow R_{c,d}} v(r)$ where $R_{c,d}$ is defined in (4.25). We then considered solutions to (1.2) where the central values $a, b > 0$ lie on the boundary ∂S .

In Theorem 4-3 we used many of our earlier results to finally establish the existence of entire large solutions to system (1.2). We showed these solutions exist for central values in E , defined in (4.32). Our more significant results for systems are summarized in the following table.

Table 5-1. Existence of Solutions for System (1.2)

$\Delta u = p(x)f(v)$ $\Delta v = q(x)g(u)$	
	Entire Domain $\Omega = \mathbb{R}^n$
Solution Exists/ Requirements	Yes b,c,d,e
Large Solution Exists/ Requirements	Yes a,c,d,f
a) $G(s) = \min_{s \leq t} \{f(t), g(t)\}$ satisfies Keller-Osserman condition (see (2.2)). b) f or g satisfies Keller-Osserman condition (see (2.2)). c) $f(0) = g(0) = 0$, $f(s) > 0$ and $g(s) > 0$ for $s > 0$. d) Functions p, q not identically zero at infinity and $\int_0^\infty tp(t) < \infty$, $\int_0^\infty tq(t) < \infty$ e) Infinitely many solutions. Central values form closed and bounded set S (see (4.1)). f) Infinitely many solutions. Central values on E (see (4.32))	

After our analysis, we identified several example systems that satisfied the requirements of our theorems and lemmas. We then examined numerical solutions for several of these examples. In addition to demonstrating the results of our analysis, our numerical results suggest the set of central values is not convex.

5.2 Further Work

While single equations similar to (1.1) have been studied extensively, there are still many open problems. Some of these potential areas of study can be seen from Table 2-1. Condition (c) from this table has been shown to be necessary for the existence of solutions. It is unknown if other conditions may be relaxed. For example, it is unknown if entire large solutions exist for the superlinear/mixed case when condition (a) does not hold. That is when

$$\int_0^\infty r \max_{|x|=r} p(x) dr = \infty \text{ or } \int_0^\infty r \max_{|x|=r} q(x) dr = \infty.$$

In fact, a similar result remains unknown for the single term equation $\Delta u = p(x)u^\alpha$. Also, the existence of entire large solutions for the sublinear case remains unknown if condition (b) of Table 2-1 is not true, that is when (1.1) is not radial. We believe that each of these problems will be extremely challenging to solve.

A much simpler problem may be to examine properties of the 2-term equation and extend them to an arbitrary n -term equation. Results will likely be similar to the 2-term problem, but this would be an interesting exercise.

There are fewer results for system (1.2) as compared to the single equation, and therefore more opportunities exist for study. Allowing f and g to be non-monotonic has created unique problems that would be worth examining further. For example, we did not characterize our set of central values beyond closed and bounded. For monotonic functions f and g , [6] and [22] both show $(a_0, b_0) \in S$ implies $(a, b) \in S$ where $0 \leq a \leq a_0$ and $0 \leq b \leq b_0$. We do not know if a similar result holds for non-monotonic functions. We have not encountered numerical examples that would suggest otherwise. However, we considered only simple systems in our numerical trials. Also, we are unsure if large solutions exist if G , as in (4.3), does not satisfy the Keller-Osserman condition (2.2). Indeed, we have shown entire solutions may still exist, but can any of these solutions be large? Finally, we would like to further relax our hypotheses. Instead of requiring that G satisfies the Keller-Osserman condition, we would like to require that only f and g need to satisfy this condition. When we began studying systems, this was our original hypothesis. However, to use our arguments, we must have a monotonic function below f and g that has all the same properties as G . Most importantly, we need the function to satisfy the Keller-Osserman condition. At this time, we believe such a construction should be possible, but

we have encountered difficulties when trying to build such a function. It would be worthwhile to examine these areas more closely.

These are just a few examples for future study. In general, this field provides many open problems, both challenging and simple, making it well suited for students and researchers of any level. Further, the wide range of application for elliptic theory allows this area of study to be valuable in a variety of fields.

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Vita

Second Lieutenant Jesse D. Peterson was born in Papillion, NE. He graduated Suma Cum Laude from South Dakota State University (SDSU) with Bachelor of Science degrees in Civil Engineering and Mathematics. He commissioned through Air Force ROTC Detachment 780 at SDSU where he received the honor of Distinguished Graduate.

Lieutenant Peterson's first active duty assignment was to attend the Air Force Institute of Technology (AFIT) to pursue a Master of Science degree in Applied Mathematics. Upon graduation from AFIT, Lieutenant Peterson was assigned to the 59th Test and Evaluation Squadron at Nellis Air Force Base, NV.

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